2008 – 2009 Log1 Contest Round 2
Theta Number Theory

Name: ___________________

this an more complex problems.		
1	If a and b are distinct digits, then how many 2-digit numbers exist such that ab and ba are prime numbers (ab means a followed by b not a times b)?	
2	How many positive factors does 378000 have?	
3	What is the least (positive) common multiple of 14, 15, and 16?	
4	Evaluate: 16 mod 5	
5	What is the base 6 representation of the base 10 number 10?	

5 points each		
6	How many distinct prime numbers are factors of 25! ?	
7	What is the sum of the 10^{th} row of Pascal's triangle if the first row is 1?	
8	How many pairs of positive integers A and B are there such that 4A+5B=85?	
9	What is the base 8 number 1225 in base 10?	
10	What is the hundreds digit of 5^{55}?	

6 points each		
11	Andy wants to go to a party at a house that's 4 blocks north and 6 blocks east of him. How many paths can Andy choose from if he can only go North and East?	
12	What is the smallest positive integer n such that 18900n is a perfect cube?	
13	What is the product of the smallest deficient number, the smallest perfect number, and the smallest abundant number?	
14	What is the sum of the first five terms in the sequence 0.1 (base2), 0.1 (base3), 0.1(base4),… Express your answer as a reduced fraction.	
15	What is the 32^{st} prime number?	

2008 – 2009 Log1 Contest Round 2
Alpha Number Theory

Name: ___________________

4 points each		
1	If a and b are distinct digits, then how many 2-digit numbers exist such that ab and ba are prime numbers (ab means a followed by b not a times b)?	
2	How many positive factors does 378000 have?	
3	What is the least (positive) common multiple of 14, 15, and 16?	
4	Evaluate: 16 mod 5	
5	What is the base 6 representation of the base 10 number 73?	

5 points each		
6	How many distinct prime numbers are factors of 25! ?	
7	What is the sum of the 10^{th} row of Pascal's triangle if the first row is 1?	
8	How many pairs of positive integers A and B are there such that 4A+5B=85?	
9	What is the base 8 number 1225 in base 10?	
10	What is the thousands digit of 25^{25}?	

6 points each		
11	Andy wants to go to a party at a house that's 4 blocks north and 6 blocks east of him. How many paths can Andy choose from if he can only go North and East?	
12	What is the smallest positive integer n such that 18900n is a perfect cube?	
13	What is the product of the smallest deficient number, the smallest perfect number, and the smallest abundant number?	
14	What is the sum of the first five terms in the sequence 0.1 (base2), 0.1 (base3), 0.1(base4),… Express your answer as a reduced fraction.	
15	I have a number of widgets. If I put them in boxes of 7 each, I have 2 left over. In boxes of 9 each, I have 7 leftover and in boxes of 13 each, there are 4 left over. What is the minimum of widgets that I have?	

2008 – 2009 Log1 Contest Round 2
Mu Number Theory

Name: ____________________

	4 points each	
1	If a and b are distinct digits, then how many 2-digit numbers exist such that ab and ba are prime numbers (ab means a followed by b not a times b)?	
2	How many positive factors does 378000 have?	
3	What is the least (positive) common multiple of 14, 15, and 16?	
4	Evaluate: 69 mod 17	
5	What is the base 6 representation of the base 10 number 73?	

	5 points each	
6	How many distinct prime numbers are factors of 25! ?	
7	What is the sum of the 10^{th} row of Pascal's triangle if the first row is 1?	
8	How many pairs of positive integers A and B are there such that 4A+5B=85?	
9	What is the remainder when 7^{105} is divided by 11?	
10	What is the thousands digit of 25^{25}?	

	6 points each	
11	Andy wants to go to a party at a house that's 4 blocks north and 6 blocks east of him. How many paths can Andy choose from if he can only go North and East?	
12	What is the smallest positive integer n such that 18900n is a perfect cube?	
13	What is the product of the smallest deficient number, the smallest perfect number, and the smallest abundant number?	
14	When the base 10 number 245! is written in base 8, how many trailing zeroes does it have?	
15	I have a number of widgets. If I put them in boxes of 7 each, I have 2 left over. In boxes of 9 each, I have 7 leftover and in boxes of 13 each, there are 4 left over. What is the minimum of widgets that I have?	

2008 – 2009 Log1 Contest Round 2
Number Theory Answers

Theta Answers	
1	8
2	160
3	1680
4	1
5	14
6	9
7	512
8	4
9	661
10	1
11	210
12	490
13	72
14	$\frac{29}{20}$
15	131

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2008 – 2009 Log1 Contest Round 2
Number Theory Solutions

Th	Al	Mu	Solution
1	1	1	Since ab and ba have to be prime numbers, a and b cannot be even or 5. This leaves 12 possible two digit numbers since the digits must be different. Trying each one yields: 13, 17, 31, 37, 71, 73, 79, and 97.
2	2	2	Prime factorizing 378000 gives $2^4 3^3 5^3 7^1$. Now finding the number of factors is a combinations problem, as every possible factor of 378000 can be rewritten as a combination of these primes. Add one to each exponent and then multiply the exponents. By adding one, we include the possibility that one or more of the prime factors have an exponent of 0. $5 \times 4 \times 4 \times 2 = 160$
3	3	3	The quickest way is to prime factorize all three numbers. $14 = 2 \times 7$ $15 = 3 \times 5$ $16 = 2^4$ Now amongst the three factorizations take the prime factors with the largest exponents and multiply them together. $2^4 \times 3 \times 5 \times 7 = 1680$
4	4		Basically asking for the remainder when 16 is divided by 5. Answer is 1.
		4	Basically asking for the remainder when 69 is divided by 17. Answer is 1.
5			To convert numbers from base 10 to base 6, the base 10 number must be rewritten in terms of powers of 6 (thus base 6) rather than powers of 10. $10 = 1(10^1) + 0(10^0) \therefore 10_{10}$ $= 1(6^1) + 4(6^0) \therefore 14_6$
	5	5	To convert numbers from base 10 to base 6, the base 10 number must be rewritten in terms of powers of 6 (thus base 6) rather than powers of 10. $73 = 7(10^1) + 3(10^0) \therefore 73_{10}$ $= 2(6^2) + 0(6^1) + 1(6^0) \therefore 201_6$
6	6	6	By definition $25! = 25 \times 24 \times 23 \times ... \times 3 \times 2 \times 1$, which displays all of the distinct prime factors of 25! in order. By disregarding all other nonprime factors, there remains 9 distinct primes left.
7	7	7	When the first row is defined as 1, we can see a pattern in the sums of each row of Pascal's triangle. The first row sums to 1, the second to 2, the third to 4, and the nth row to 2^{n-1}. Therefore the 10[th] row sums to $2^{(10)-1} = 2^9 = 512$. Pascal's Triangle, Left Justified 1 1 1 1 2 1 2 1 = 4 1 3 3 1 8 1 4 6 4 1 16
8	8	8	If B=1, then A=20. Since 4 and 5 are relatively prime, the next possibility is B=1+4=5. Other possibilities are B=9 and 13. B=17 does not work as then A would be 0; so there are 4 pairs in total.
9	9		A base 8 number is simply a base 10 number rewritten using powers of 8 rather than 10. $1225_8 = 1(8^3) + 2(8^2) + 2(8^1) + 5(8^0)$ $= 1(512) + 2(64) + 2(8) + 5(1)$ $= 661_{10}$

<table>
<tr><td></td><td></td><td>9</td><td>It is clear that 7^2 has a remainder of 5 when divided by 11. So, 7^3 has the same remainder as 7x5 or 2 when divided by 11. 7^5 has the same remainder as 2x5=10 or the same as -1. Therefore, $7^{105} = (7^5)^{15}$ has the same remainder of $(-1)^{15}=-1$ and the remainder is 10. Using properties of congruences mod 11 is a good way to show this.</td></tr>
<tr><td>10</td><td></td><td></td><td>Sampling the hundreds digit for the first few powers of 5 yields:
$5^1 = 5 \therefore 0$
$5^2 = 25 \therefore 0$
$5^3 = 125 \therefore 1$
$5^4 = 625 \therefore 6$
$5^5 = ...125 \therefore 1$
This pattern of 1s and 6s continues to repeat, allowing us to determine the hundred's digit of 5^{55}. Now taking the exponent we subtracting 2 to account for the fact that the pattern doesn't begin until 5^3. Using modular arithmetic ($53 \bmod 2 = 1$) we obtain that 5^{55} corresponds to the first term, which has a hundreds digit of 1.</td></tr>
<tr><td></td><td>10</td><td>10</td><td>Sampling the hundreds digit for the first few powers of 25 yields:
$25^1 = 25 \therefore 0$
$25^2 = 625 \therefore 0$
$25^3 = ...5625 \therefore 5$
$25^4 = ...0625 \therefore 0$
$25^5 = ...5625 \therefore 5$
This pattern of 5s and 0s continues to repeat, allowing us to determine the hundred's digit of 25^{25}. Now taking the exponent we subtracting 2 to account for the fact that the pattern doesn't begin until 25^3. Using modular arithmetic ($23 \bmod 2 = 1$) we obtain that 25^{25} corresponds to the first term, which has a thousands digit of 5.</td></tr>
<tr><td>11</td><td>11</td><td>11</td><td>Constructing a 4x6 array, we see that there are many ways to travel 4 blocks North and 6 blocks East. As each intersection point in the array combines the number of paths from those intersection points located directly South and West, Pascal's Triangle can be applied, as shown below:
<table>
<tr><td>1</td><td>5</td><td>15</td><td>35</td><td>70</td><td>126</td><td>210</td></tr>
<tr><td>1</td><td>4</td><td>10</td><td>20</td><td>35</td><td>56</td><td>84</td></tr>
<tr><td>1</td><td>3</td><td>6</td><td>10</td><td>15</td><td>21</td><td>28</td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>
Each number represents the number of paths to one can take to reach the intersection point located Northeast of the number. Another way to look at it is that of the 10 steps to get there, any 4 of them will be north and the other 6 east so there are 10C4=210 ways.</td></tr>
<tr><td>12</td><td>12</td><td>12</td><td>In order for 18900n to be a perfect cube each individual prime factor in the prime factorization of 18900n has to be a perfect cube as well, in other words the exponent of the primes in the factorization have to be divisible by 3.
$18900n = 2^2 \times 3^3 \times 5^2 \times 7 \times n$
$\therefore n = 2^1 \times 5^1 \times 7^2 = 490$</td></tr>
<tr><td>13</td><td>13</td><td>13</td><td>A deficient number is an integer whose summation of its factors is less than twice the original number. A perfect number is an integer whose summation of its factors is equal to twice the original number. An abundant number is an integer whose summation of its factors is more than twice the original number. Therefore the smallest deficient, perfect, and abundant numbers are 1, 6, and 12, respectively.
$1 \times 6 \times 12 = 72$</td></tr>
</table>

14	14		The sequence can be rewritten in base 10 fractions: $0.1_2 + 0.1_3 + 0.1_4 + 0.1_5 + 0.1_6$ $= \left(2^{-1}\right) + \left(3^{-1}\right) + \left(4^{-1}\right) + \left(5^{-1}\right) + \left(6^{-1}\right)$ $= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ $= \frac{87}{60} = \frac{29}{20}$
		14	An easy method to finding the number of zeros in base 8 is to find the number of trailing zeros in base 2, or the number of powers of 2 in 245!, by repeatedly dividing 245 and resultant quotients by 2, and summing the integral values of the quotients: $2\overline{)245} = 122R1$ $2\overline{)122} = 61R0$ $2\overline{)61} = 30R1$ $2\overline{)30} = 15R0$ $2\overline{)15} = 7R1$ $2\overline{)7} = 3R1$ $2\overline{)3} = 1R1$ $2\overline{)1} = 0R1$ $\therefore 122 + 61 + 30 + 15 + 7 + 3 + 1 = 239$ There are 239 trailing zeros in base 2, which translates to 79 trailing zeros in base 8 since one needs three 2's to make a factor of 8.
15			Brute Force. Go through each number one by one until you have obtained the 32nd prime number. It may have been helpful to know that there are 25 primes less than 100. Up to 121, one only has to check 2, 3, 5 and 7. After that you only to include 11 as well.
	15	15	One can start with the first condition that the number have a remainder of 2 when divided by 7. Listing these we get 2, 9, 16, 23, … We see that 16 meets the condition that it have a remainder of 7 when divided by 9. Unfortunately, the remainder when divided by 13 is 3 (not 4). Now the next number that meets the first two conditions is 16+63=79 since 7 and 9 are relative prime. Since 63 has a remainder of 11 when divided by 13, we can get the new remainder after adding 63 by adding 11 (or subtracting 2) to the previous remainder. Since 3 – 2 + 11 – 2 – 2 – 2 – 2 = 4, we have to add 6x63 to 16 getting 394 as the smallest number meeting the three conditions. The Chinese Remainder Theorem can also be used to solve this an more complex problems.