1. B
2. A
3. A
4. A
5. D
6. D
7. C
8. B
9. B
10. C
11. C
12. B
13. C
14. B
15. A
16. D
17. C
18. A
19. D
20. B
21. D
22. C
23. D
24. B
25. C
26. A
27. C
28. C
29. A
30. D

1. B Let the third side length be x. By the triangle inequality, we have $x + 20 > 22$, $x + 22 > 20$, and $20 + 22 > x$. The intersection of these gives $2 < x < 42$, which gives $\boxed{39}$ possible values of x.

2. A This is a right triangle with legs of length 20 and 22, so the area is $\frac{1}{2} \cdot 20 \cdot 22 = \boxed{220}$.

3. A We compute $5^2 + 6^2 = 61 < 64 = 8^2$, so the triangle is $\boxed{\text{obtuse}}$.

4. A The two triangles are similar with a scale factor of 2, so the area of the second triangle is $2^2 = 4$ times the area of the first triangle, which is $4 \cdot 6 = \boxed{24}$.

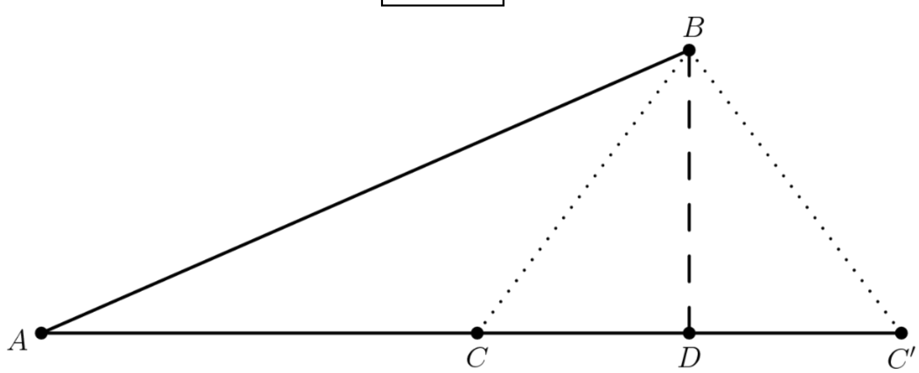
5. D Let the degree measures of the angles be $m - d$, m, and $m + d$. We have that $(m - d) + m + (m + d) = 180° \Rightarrow m = \boxed{60°}$

6. D Using the formula $K = \frac{s^2\sqrt{3}}{4}$ with $s = 6$ gives an area of $\boxed{9\sqrt{3}}$

7. C Let M be the midpoint of $\overline{AB}$, and let N be the midpoint of $\overline{AC}$. ΔAMN is similar (by SAS) to ΔABC with a scale factor of $\frac{1}{2}$, so $MN = \frac{1}{2} \cdot BC = \boxed{\frac{5}{2}}$.

8. B Letting the exterior angles be $4x$, $5x$, and $6x$, we have that $4x + 5x + 6x = 360°$ so $x = 24°$. This gives exterior angles of $96°$, $120°$, and $144°$, which correspond to interior angles of $180° - 96° = 84°$, $180° - 120° = 60°$, and $180° - 144° = 36°$. This gives a ratio of $84 : 60 : 36 = \boxed{7 : 5 : 3}$.

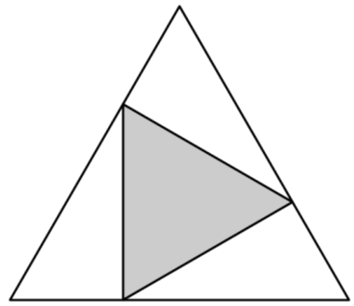
9. B



Let D denote the foot of the perpendicular from BD to line AC. We can compute that $BD = AB \cdot \sin(A) = 10 \cdot \frac{2}{5} = 4$. By the Pythagorean theorem, $CD = \sqrt{BC^2 - BD^2} = \sqrt{5^2 - 4^2} = 3$. The two possible locations of vertex C are 3 units from D in opposite directions, so the distance between them is $\boxed{6}$.

10. C Note that a right triangle with a right angle at B, $AB = 1$, and $BC = \sqrt{2}$ satisfies the given constraints. The hypotenuse of such a triangle is $\sqrt{3}$ by the Pythagorean theorem, so we can compute that $\sin(C) = \frac{1}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{3}}$.

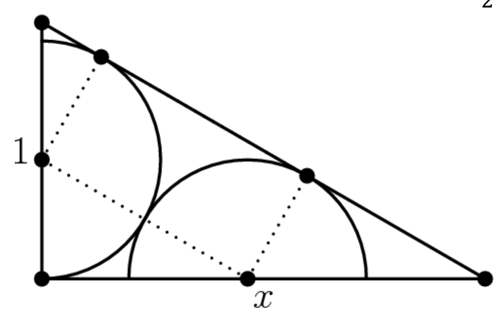
11. C



Let the side length ΔABC be s. Its area is given by $\frac{s^2\sqrt{3}}{4}$. Each of the small white triangles has area given by $\frac{1}{2}\cdot\frac{s}{3}\cdot\frac{2s}{3}\cdot\sin(60°)=\frac{s^2\sqrt{3}}{18}$. The area of ΔDEF is thus $\frac{s^2\sqrt{3}}{4}-3\cdot\frac{s^2\sqrt{3}}{18}=\frac{s^2\sqrt{3}}{12}$. Our desired ratio is thus $\frac{s^2\sqrt{3}}{12}/\frac{s^2\sqrt{3}}{4}=\boxed{\frac{1}{3}}$

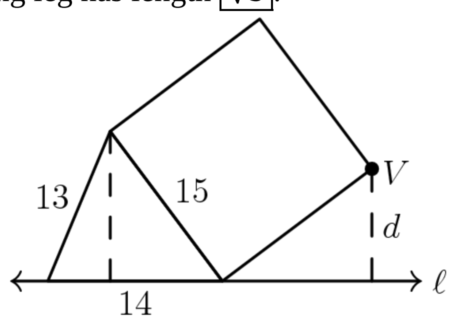
12. B Note that the base and height of the shaded triangle are equal to the side length of the smallest equilateral triangles and the altitude length of the labeled equilateral triangle respectively. The three smallest equilateral triangles have half the side length of the labeled one, so the area of the shaded triangle ends up being $\frac{1}{2}\cdot 12=\boxed{6}$

13. C



Let the radius of the two circles be r. Connect the centers of the semicircles to each other and to the points of tangency on the hypotenuse of the triangle to split the triangle into a rectangle and three smaller right triangles. The bottom left right triangle has a right angle, a leg of length r, and a hypotenuse of length $2r$, so it is a 30-60-90 triangle. Some simple angle chasing shows that the big triangle is similar to this triangle, so the long leg has length $\boxed{\sqrt{3}}$.

14. B



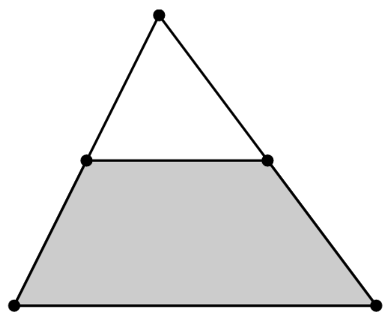
By Heron's formula, the area of the 13-14-15 triangle is $\sqrt{21(8)(7)(6)}=84$. Dropping an altitude to the side of length 14, we can see that it has length $\frac{2\cdot 84}{14}=12$.

Some simple angle chasing yields that the two right triangles with hypotenuse 15 are congruent, so $d = \sqrt{15^2 - 12^2} = \boxed{9}$.

15. A Note that the first two squares have side lengths 7 and 4, respectively. This creates a 3-4-5 right triangle above the second square. Let the side length of the third square be s. Since the diagram is to scale and the triangle is isosceles, the side of the triangle contained within the third square is also equal to 5 (we don't actually need to use the fact that the diagram is to scale, since the other two possible configurations of isosceles triangles are impossible (prove it!), but it makes our lives easier). By the Pythagorean theorem, we can say that $s^2 + (s-4)^2 = 25$, so $2s^2 - 8s - 9 = 0$. Solving this, we get $s = 2 \pm \frac{\sqrt{34}}{2}$. s must be positive, so $s = 2 + \frac{\sqrt{34}}{2}$. x is the area of the square, which is $\left(2 + \frac{\sqrt{34}}{2}\right)^2 = \boxed{\frac{25}{2} + 2\sqrt{34}}$.

16. D $y = x$ and $y = -x$ intersect at $(0,0)$, and the line $y = 2x - 6$ intersects $y = x$ and $y = -x$ at $(6,6)$ and $(2,-2)$, respectively, which are distances $6\sqrt{2}$ and $2\sqrt{2}$ from $(0,0)$, respectively. Since $y = x$ is perpendicular to $y = -x$, the desired area is simply $\frac{1}{2}(6\sqrt{2})(2\sqrt{2}) = \boxed{12}$.

17. C

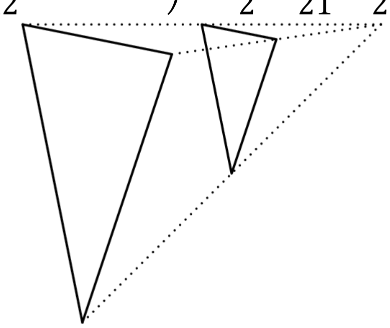


Note that ΔABC and ΔAXY are similar (by SAS) with a similarity ratio of 2. This means if the area of ΔAXY is K, then the area of ΔABC is $4K$ and thus the area of quadrilateral $BXYC$ is $3K$. This gives $3K = 30 \Longrightarrow K = 10$, so the area of ΔABC is $4(10) = \boxed{40}$.

18. A Let $x = CD$ and h be the common height of $ABCD$ and ABP. We have that

$$\frac{1}{2} \cdot 1 \cdot h = \frac{20}{21}\left(\frac{1}{2}(1+x) \cdot h\right) \Longrightarrow \frac{1}{2} = \frac{10}{21} + \frac{10}{21}x \Longrightarrow x = \boxed{\frac{1}{20}}.$$

19. D



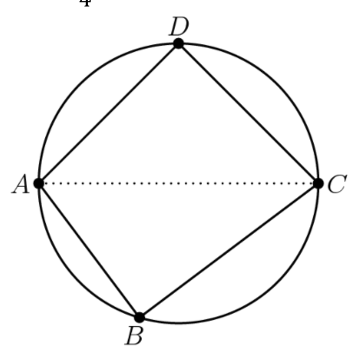
Note that $DE = \frac{1}{2}AB, EF = \frac{1}{2}BC$, and $DF = \frac{1}{2}AC$, so ΔABC and ΔDEF are similar with similarity ratio $\frac{1}{2}$. This means the area of ΔDEF is $\frac{1}{4}$ the area of ΔABC. The area of ΔABC can be calculated using the shoelace formula:

$$\begin{matrix} -2 & 6 \\ 3 & 5 \\ 0 & -4 \\ -2 & 6 \end{matrix}$$

$$K = \frac{1}{2}|(-2 \cdot 5 + 3 \cdot -4 + 0 \cdot 6) - (3 \cdot 6 + 0 \cdot 5 + -2 \cdot -4)| = 24.$$

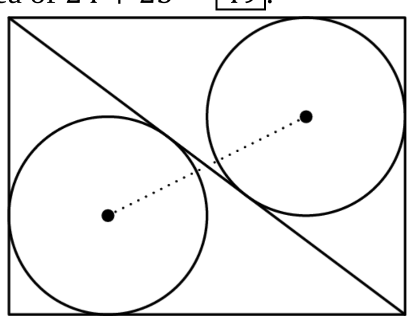
Which means the area of ΔDEF is $\frac{1}{4} \cdot 24 = \boxed{6}$.

20. B



Draw diagonal $\overline{AC}$, splitting $ABCD$ into ΔABC and ΔADC. Since angle B is right, the area of ΔABC is $\frac{1}{2}(6)(8) = 24$. We also know that $\overline{AC}$ is a diameter of the circle because angle B is right. To maximize the area of ΔADC, we place D as far away from $\overline{AC}$ as possible, which is the midpoint of semicircle AC. This gives an isosceles right triangle, the circumradius of which is $\frac{1}{2}\sqrt{6^2 + 8^2} = 5$. This gives the area of ΔADC as 25, for a total area of $24 + 25 = \boxed{49}$.

21. D

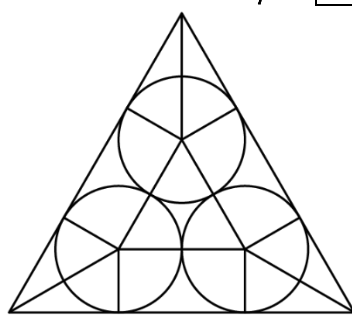


Triangles ABC and ADC are 3-4-5 right triangles. We can easily compute their areas to be $K = \frac{1}{2}(3)(4) = 6$ and semiperimeters to be $\frac{1}{2}(3 + 4 + 5) = 6$. This means their inradii are given by $r = \frac{K}{s} = \frac{6}{6} = 1$. This means the centers of the circles are separated by $4 - 2(1) = 2$ in the horizontal direction and $3 - 2(1) = 1$ in the vertical direction, for a total distance of $\sqrt{2^2 + 1^2} = \boxed{\sqrt{5}}$

22. C Let θ be the angle between the sides of lengths 7 and 10. We have that $28 = \frac{1}{2} \cdot 7 \cdot 10 \cdot \sin(\theta)$. Rearranging, we can find that $\sin(\theta) = \frac{4}{5}$. Since θ is acute, we can draw a 3-4-5 right triangle to see that $\cos(\theta) = \frac{3}{5}$. By the law of cosines, we can compute that the third side of the triangle has length $\sqrt{7^2 + 10^2 - 2 \cdot 7 \cdot 10 \cdot \frac{3}{5}} = \boxed{\sqrt{65}}$.

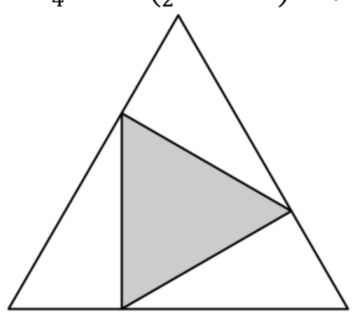
23. D Let $m\angle A = 2\alpha$ and $m\angle C = 2\gamma$, and note that $40^\circ + 2\alpha + 2\gamma = 180^\circ \Longrightarrow \alpha + \gamma = 70^\circ$. The incenter is the intersection of angle bisectors, so $m\angle IAC = \alpha$ and $m\angle ICA = \gamma$. Therefore, $m\angle AIC = 180^\circ - \alpha - \gamma = \boxed{110^\circ}$

24. B



The equilateral triangle can be partitioned into three rectangles with length 2 and height 1, an equilateral triangle of side length 2, and six 1-$\sqrt{3}$-2 right triangles. This gives a total area of $3(2 \cdot 1) + \frac{2^2\sqrt{3}}{4} + 6\left(\frac{1}{2} \cdot \sqrt{3} \cdot 1\right) = \boxed{6 + 4\sqrt{3}}$.

25. C



Let the side length ΔABC be s. Its area is given by $\frac{s^2\sqrt{3}}{4}$. Each of the small white triangles has area given by $\frac{1}{2} \cdot \frac{s}{3} \cdot \frac{2s}{3} \cdot \sin(60^\circ) = \frac{s^2\sqrt{3}}{18}$. The area of ΔDEF is thus $\frac{s^2\sqrt{3}}{4} - 3 \cdot \frac{s^2\sqrt{3}}{18} = \frac{s^2\sqrt{3}}{12}$. Our desired ratio is thus $\frac{s^2\sqrt{3}}{12} / \frac{s^2\sqrt{3}}{4} = \boxed{\frac{1}{3}}$

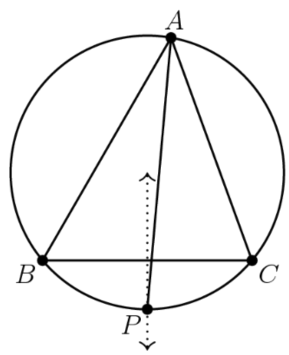
26. A We proceed by casework on the largest angle, which we will call α for brevity. First note that $60^\circ \leq \alpha \leq 160^\circ$. Let the second largest angle be β.

- If $\alpha = 160^\circ$, β can only be 10°, which gives 1 triangle.
- If $\alpha = 150^\circ$, β can only be 20°, which gives 1 triangle.
- $\alpha = 140^\circ \Longrightarrow \beta \in \{20^\circ, 30^\circ\} \Longrightarrow$ 2 triangles.
- $\alpha = 130^\circ \Longrightarrow \beta \in \{30^\circ, 40^\circ\} \Longrightarrow$ 2 triangles.
- $\alpha = 120^\circ \Longrightarrow \beta \in \{30^\circ, 40^\circ, 50^\circ\} \Longrightarrow$ 3 triangles.

- $\alpha = 110° \Longrightarrow \beta \in \{40°, 50°, 60°\} \Longrightarrow 3$ triangles.
- $\alpha = 100° \Longrightarrow \beta \in \{40°, 50°, 60°, 70°\} \Longrightarrow 4$ triangles.
- $\alpha = 90° \Longrightarrow \beta \in \{50°, 60°, 70°, 80°\} \Longrightarrow 4$ triangles.
- $\alpha = 80° \Longrightarrow \beta \in \{50°, 60°, 70°, 80°\} \Longrightarrow 4$ triangles.
- $\alpha = 70° \Longrightarrow \beta \in \{60°, 70°\} \Longrightarrow 2$ triangles.
- $\alpha = 60° \Longrightarrow \beta = 60° \Longrightarrow 1$ triangle.

This gives a final answer of $1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 4 + 2 + 1 = \boxed{27}$ triangles.

27. C



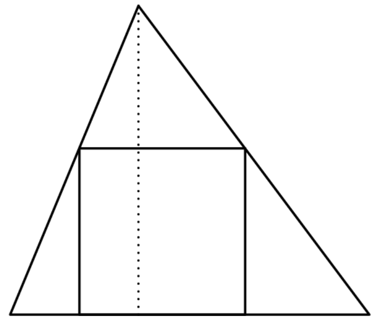
Draw in the circumcircle of ΔABC, and note that the perpendicular bisector of $\overline{BC}$ intersects the circumcircle at the midpoint of minor arc BC. Line AD bisects angle A, so as a consequence of the inscribed angle theorem, it also bisects minor arc BC and thus also intersects it at its midpoint, so P lies on the circumcircle. By the inscribed angle theorem again, it follows that $m\angle BPC = m\angle B + m\angle C = 60° + 70° = \boxed{130°}$

28. C Call the three points A, B, and C. Choose A arbitrarily and draw a diameter of the circle through A that divides the circle into two semicircles. The angle at C is obtuse if and only if:

- B and C lie on the same semicircle.
- C is between A and B on that semicircle.

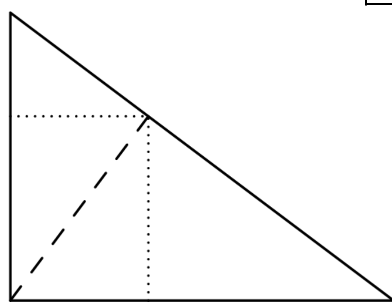
The probability of the first bullet point is clearly $\frac{1}{2}$. Given that B and C lie on the same semicircle, each configuration that has C between A and B can be matched up one-to-one with a configuration that has B between A and C by switching B and C, so the probability of the second bullet point is also $\frac{1}{2}$. This gives a total probability of $\frac{1}{4}$ that angle C is obtuse. The same reasoning can be applied to the other two vertices, and since at most one angle can be obtuse, our events are mutually exclusive, and the probability of an acute triangle is $3 \cdot \frac{1}{4} = \boxed{\frac{3}{4}}$.

29. A



Let the side length of the square be s. Note that the top triangle is similar to the large triangle due to the parallel sides of the square. We can compute that the altitude to the side of length 14 has length 12 (see problem 14), so we can set up the similarity ratio $\frac{14}{s} = \frac{12}{12-s}$. Solving this yields $26s = 168 \Longrightarrow s = \boxed{\frac{84}{13}}$.

30. D



Note that all of the created right triangles are similar; as a result, the length of the altitude to the hypotenuse of a given triangle is proportional to the length of the hypotenuse of that triangle. The altitude for S_1 corresponds to the hypotenuse of length 5, while the altitudes for S_2 correspond to hypotenuses of lengths 3 and 4, so their sum corresponds to a sum of 7. This gives $S_2 = \frac{7}{5}S_1$. Applying this same reasoning to each of the smaller triangles, we can find that $S_3 = \frac{7}{5}S_2$, and in general, $S_{n+1} = \frac{7}{5}S_n$. This means that $S_{22} = \frac{7}{5}S_{21} = \frac{7}{5}\left(\frac{7}{5}S_{20}\right) = \frac{49}{25}S_{20}$, which gives a final ratio $\frac{S_{20}}{S_{22}} = \boxed{\frac{25}{49}}$.