Alpha

Matrices and Vectors

Test #122

Directions:

1. Fill out the top left section of the scantron. Do not abbreviate your school name.

2. In the Student ID Number grid, write your 9-digit ID# and bubble.

3. In the Test Code grid, write the 3-digit test# on this test cover and bubble.

4. Scoring for this test is 5 times the number correct plus the number omitted.

5. TURN OFF ALL CELL PHONES.

6. No calculators may be used on this test.

7. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

8. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

9. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

10. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

11. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

12. Unless a question asks for an approximation or a rounded answer, give the exact answer.

NOTA means none of the above. Unless otherwise stated, $\boldsymbol{i} = \langle 1,0,0\rangle, \boldsymbol{j} = \langle 0,1,0\rangle, \boldsymbol{k} = \langle 0,0,1\rangle$.

1. Consider the system of equations

$$-2x + 3y = 5$$
$$4x + ry = s$$

where $r, s \in \mathbb{R}$. If the system has infinitely many solutions, then the product of r and s is

A. -36 B. 36 C. 60 D. 72 E. NOTA

2. How many of the following matrices are product of **two** elementary matrices?

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$$

A. 1 B. 2 C. 3 D. 4 E. NOTA

3. For matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Find A^{2020}

A. $2^{1010} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ B. $2^{2020} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

C. $-2^{1010} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ D. $-2^{2020} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ E. NOTA

4. For all 2×2 matrices A, B, consider the statement $(A + B)^2 = A^2 + 2AB + B^2$. Which of the following is the most accurate?

A. The statement is true by distributive and associative properties.
B. The statement is true by distributive and commutative properties.
C. The statement is false because the associative property does not hold.
D. The statement is false because the commutative property does not hold.
E. NOTA

5. A matrix M is *nilpotent* if there exists some positive integer n such that M^n is the zero matrix. How many of the following matrices are nilpotent?

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -6 & -4 \\ 9 & 6 \end{bmatrix}$$

A. 1 B. 2 C. 3 D. 4 E. NOTA

6. Suppose A, B are $n \times n$ matrices such that A is nilpotent, and B is invertible. How many of the following statements are always true?

I. AB is invertible
II. AB is nilpotent
III. $\det(AB) = \det(B)$
IV $rank(AB) = n$

A. 1 B. 2 C. 3 D. 4 E. NOTA

7. Consider the inverse of the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ * & -1 & -1 \\ * & * & -1 \end{bmatrix}$$

Compute the sum of the missing entries.

A. -6 B. -4 C. 0 D. 4 E. NOTA

8. In continuation from the previous problem, find the inverse of $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & -9 \\ 0 & 1 & -1 \end{bmatrix}$

A. $\begin{bmatrix} -3 & -2 & -1 \\ -1 & -1 & -1 \\ 4 & 3 & 3 \end{bmatrix}$ B. $\begin{bmatrix} -3 & 8 & -1 \\ -1 & -1 & -1 \\ 4 & 3 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 4 & -1 & -3 \\ 3 & -1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$ D. $\begin{bmatrix} 4 & -1 & -3 \\ 3 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}$ E. NOTA

9. Given matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 4 & 5 & 0 \\ 6 & 7 & 0 & 0 \end{bmatrix}$, find the determinant of $\left(\frac{1}{2}A\right)^3$.

A. 27 B. 216 C. 1728 D. 13824 E. NOTA

10. Suppose 4×4 matrix A has determinant 5. The following operations are performed in sequence to obtain matrix B:
    - Exchange the second and the third row.
    - Multiply each entry of the first column by 4
    - Add three times the first row to the third row.
    - Take the third power.
    - Divide each entry by 2.

    Find the determinant of B.

    A. -13500 B. -4000 C. -500 D. 4000 E. NOTA

11. The 3×3 matrix A has entries drawn from the set $\{0, \pm 1, \pm 2, \pm x, \pm y, \pm z\}$. Let vector $v = \langle x, y, z \rangle$. If $Av = \langle z^2, y^2, x^2 \rangle$, find the determinant of A.

    A. $x + y + z$ B. $-xyz$ C. $xy + yz - xz$ D. Not enough information E. NOTA

12. For how many values of s are the vectors $\langle 1, s, 1, s \rangle$ and $\langle s, 1, s, 1 \rangle$ parallel?

    A. 0 B. 1 C. 2 D. ∞ E. NOTA

13. For how many values of s are the vectors $\langle 1, s, s, 2s \rangle$ and $\langle 4, s, s^3, -3s \rangle$ orthogonal?

    A. 0 B. 1 C. 2 D. ∞ E. NOTA

14. Find the product of all values s such that the vector $\langle -3, -2, -1, s \rangle$ has length 5.

    A. -11 B. -9 C. -8 D. -7 E. NOTA

15. Find the sum of all values s such that the vector $\langle s - 5, -2, 2, s \rangle$ has length 5.

    A. -17 B. $3\sqrt{5}$ C. $-2\sqrt{3}$ D. $-3\sqrt{2}$ E. NOTA

16. For vector $u \in \mathbb{R}^3, a \in \mathbb{R}$, how many of the following expressions are formed correctly?

$$u + au\,, \quad u + u \cdot u\,, \quad u + u \times u\,, \quad a + u \cdot u$$

A. 1 B. 2 C. 3 D. 4 E. NOTA

17. Consider the sequence $a_0 = \boldsymbol{k}, a_{n+1} = \boldsymbol{i} \times a_n$ for $n \geq 0$. Which is a_{2020}?

A. $\boldsymbol{j}$ B. $-\boldsymbol{j}$ C. $\boldsymbol{k}$ D. $-\boldsymbol{k}$ E. NOTA

18. A tetrahedron has vertices at $(-3, 2, 1), (2, 2, 2), (5, -1, 2), (6, 2, -2)$. Find the volume of the tetrahedron.

A. 12 B. 24 C. 36 D. 72 E. NOTA

19. Consider three points in $\mathbb{R}^3$: $(1, 1, 0), (1, 0, 1), (0, 1,1)$. Which of the following best describes the three points?

A. The points lie on a line, and the line does not pass through the origin.
B. The points lie on a line, and the line passes through the origin.
C. The points do not lie on a line, and the plane containing the points does not pass through the origin.
D. The points do not lie on a line, and the plane containing the points passes through the origin.
E. NOTA

20. Consider the planes $x - 2y + z = 4$ and $2x + 3y - z = 7$. Find the tangent of the dihedral angle between them.

A. $-\frac{5\sqrt{59}}{59}$ B. $-\frac{\sqrt{59}}{5}$ C. $\frac{\sqrt{59}}{5}$ D. $\frac{5\sqrt{59}}{59}$ E. NOTA

21. Which point on the sphere $x^2 + y^2 + z^2 - 8x - 10y - 12z + 68 = 0$ is closest to the plane $x + 2y + 2z = 4$.

A. $(2, 4, 4)$ B. $(2, 3, 5)$ C. $(3, 3, 4)$ D. $(2, 3, 4)$ E. NOTA

22. For how many values of s are the vectors $\langle s, s+1, 1, 0\rangle, \langle s, 2s, 0, s\rangle, \langle 0, 1-s, -s, 1\rangle$ linearly dependent?

A. 0 B. 1 C. 3 D. ∞ E. NOTA

23. A particle is at position $(4, 5, 5)$ at time $t = 0$. Its velocity vector is always $\langle 1,2,2\rangle$. Find its displacement during its travels for the first 2 time units.

A. $\langle 2,4,4\rangle$ B. $\langle 6,9,9\rangle$ C. 6 D. $\sqrt{198}$ E. NOTA

24. An airline offers the following flights between Miami (MIA), Jacksonville (JAX), and Pensacola (PEN), including a scenic flight from Miami to Miami along the Biscayne Bay and the Keys: MIA→JAX, JAX→PEN, PEN→JAX, PEN→MIA, MIA→MIA. How many ways can a traveler go from Miami to Pensacola by taking exactly 8 flights offered by the airline?

A. 2 B. 8 C. 13 D. 21 E. NOTA

25. Find the distance between the planes $x + 2y + 2z = 6$ and $x + 2y + 2z = 36$

A. 6 B. 10 C. 15 D. 30 E. NOTA

26. Consider the triangle with vertices $(0, 2), (-\sqrt{3}, -1), (\sqrt{3}, -1)$. How many of the following matrices, via left multiplication, map the triangle to itself?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

A. 1 B. 2 C. 3 D. 4 E. NOTA

27. Consider the same triangle above. How many distinct invertible 2×2 matrices, via left multiplication, map the triangle to itself?

A. 3 B. 6 C. 12 D. ∞ E. NOTA

28. Consider the cube in $\mathbb{R}^3$ with vertices on $(\pm 1, \pm 1, \pm 1)$. How many of the following operations map the cube to itself?

I. Reflection on the xy plane.
II. Rotation about the z-axis by $90°$
III. Reflection on the plane $x = y$
IV. Rotation about the line $x = y = z$ by $120°$

A. 1 B. 2 C. 3 D. 4 E. NOTA

29. Consider the same cube above. How many distinct invertible 3×3 matrices, via left multiplication, map the cube to itself?

A. 24 B. 36 C. 48 D. ∞ E. NOTA

30. Which of the following is not an eigenvector of $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$?

A. $\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -6 \\ -4 \\ 2 \end{bmatrix}$ C. $\begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$ E. NOTA