1. C
2. C
3. D
4. E
5. D
6. B
7. B
8. B
9. A
10. D
11. B
12. A
13. B
14. A
15. B
16. D
17. C
18. D
19. A
20. D
21. C
22. A
23. A
24. A
25. C
26. A
27. C
28. B
29. B
30. A

Solutions:

1. C. We can rewrite the limit as $\lim_{x\to 0} e^{\cot x \ln(1+x)} = e^{\frac{\ln(1+x)}{\tan x}}$, and by L'Hopital on the exponent, we find that it has limit 1, which implies that the answer is e.

2. C. Factoring, we get that the expression in the limit is actually $(1 + x + \cdots + x^{2020})^2$ which has limit at $x = 1$ of 2021^2.

3. D. Using the first two terms of the Taylor Polynomial at $x = 0$ will suffice, and gives us $\lim_{x\to 0} \frac{x - \frac{x^3}{3} + O(x^5) - x}{x - \frac{x^3}{6} + O(x^5) - x} = 2$.

4. E. Since we do not know about the higher order derivatives of f (which is dependent on g), we cannot simply apply L'Hopital one time to get an answer, and we cannot guarantee any of the choices are correct.

5. D. We check that at $x = 0$ the numerator has value 1 and the denominator is 0, so the limit does not exist at $x = 0$.

6. B. Note that as a gets large, the largest root of $x^3 - 4ax + 1$ gets close to the largest root of $x^3 - 4ax$, which has its only positive solution at $x = 2\sqrt{a}$. For this limit to exist with the properties, we need that $p = \frac{1}{2}$ and $k = 2$, so $pk = 1$.

7. B. As the number of subintervals grows large, the trapezoidal approximation will get close to the actual area under the curve, which is $\int_0^\pi x^2 \sin x \, dx = \pi^2 - 4$.

8. B. We evaluate the integral then the limit: $\int_0^1 nx^n \, dx = \frac{n}{n+1}$ and $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Note that we cannot use the opposite order, since the sequence of functions $f_n(x)$ is pointwise and not uniformly convergent, so the achieved answers are not necessarily equal.

9. A. We need to compute the minimum value of the first derivative so we can use the C term to adjust it to always be positive. Note that $\frac{d^2}{dx^2} \frac{e^x}{e^x+1} = \frac{d}{dx} \frac{e^x}{(e^x+1)^2} = \frac{e^x(e^x-1)}{(e^x+1)^3}$ which is minimized at $x = 0$ and has first derivative value of $\frac{1}{4}$. Therefore, as long as $C \geq \frac{1}{4}$ this condition is satisfied.

10. D. We start by finding that the second derivative is $\frac{d^2f}{dx^2} = -12\sin 2x - 16\cos 2x + 2C$, and since the amplitude of the linear sine and cosine combination part is $\sqrt{12^2+16^2} = 20$, we need that $C \geq 10$ for the function to always be convex.

11. B. We can compute the first and second derivatives to be $f'(e) = 0$ and $f''(e) = -\frac{1}{e^2}$, which, by the Second Derivative Test, makes this a local max.

12. A. We see that $f'(0) = 0$ and $f''(0) = 2$ from the coefficients of the Taylor Polynomial, so by the Second Derivative Test, $f(0)$ is a local minimum.

13. B. Evaluate the integral to get $\frac{d}{dx}x^2 = 2x = 2$ at $x = 1$.

14. A. $\log f(x) = (1+x^2)\log x \Rightarrow \frac{f'(x)}{f(x)} = 2x\log x + \frac{1+x^2}{x}$. Since $f(1) = 1$, we have that $f'(1) = 2$.

15. B. $\cos(x+y)\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx} \Rightarrow -1 - \frac{dy}{dx} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$.

16. D. Considering the graphs of $f(x) = \{x\}$ and $g(x) = [x]$, we see that $f'(x) = 1$ for all noninteger x and $g'(x) = 0$ for all noninteger x. Using this and employing the chain rule, we see that $\frac{d}{dx}\{x\}^2[x]^2 = 2\{x\}[x]^2(1) + 2\{x\}^2[x](0) = 2(0.022)(4) = 0.176$.

17. C. Applying the MVT gives $\frac{b^3-a^3}{b-a} = b^2 + ab + a^2 = 13$. Consider the choices, we see that $(a,b) = (1,3)$ is a valid interval.

18. D. $f'(x) = 3x^2 - 2kx + 1$. In order for there to be no turning points, we need this quadratic to have no solutions, so using the discriminant, we see that $4k^2 < 12 \Rightarrow k^2 < 3$ for the discriminant to be negative. This makes $-1,0,1$ the only integer solutions.

19. A. Note that $f(2) = 0$ and $f'(2) = 0$, so the tangent line approximation is $y = 0$.

20. D. Using $f(2) = -1$ and $f'(2) = 7$, we get that $2 - \frac{-1}{7} = \frac{15}{7}$.

21. C. We compute $\frac{dy}{dx} = -\frac{x}{y}$, which makes the tangent line $3x + 4y = c$. Plugging in the point, we see that the tangent line is exactly $3x + 4y = 5$. After verifying the point of intersection exists, we see that $3m + 4n = 5$ because it is on the line.

22. A. To be tangent at a point, the graphs must have the same value and same first derivative. This gives us the system $a+b+1=6$ and $2+a=7$, which has solution $(a,b)=(5,0)$. Since it is tangent at $x=1$, we know that $x^3+4x+1-(x^2+5x)$ has a double root at $x=1$, and it factors as $(x-1)^2(x+1)$ so there is only one bounded region. We then integrate $\int_{-1}^{1}(x-1)^2(x+1)\,dx=\frac{4}{3}$.

23. A. If $\sin^5 x+\cos^5 x=-1$, then either $\sin x=-1$ or $\cos x=-1$. If the latter then $\sin 5x=0$, if the former, then $\sin 5x=\sin\left(5\left(2\pi k+\frac{3\pi}{2}\right)\right)=\sin\frac{3\pi}{2}=-1$.

24. A. To find the max, we differentiate once and set equal to 0, and we find that the max occurs where $1-5x=0\Rightarrow x=\frac{1}{5}$.

25. C. We will consider $f(x)=x^{-\frac{1}{3x}}$. Taking the log and differentiating gives $\frac{f'(x)}{f(x)}=\frac{-3+3\ln x}{9x^2}$ which is 0 when $x=e$, which would be the global minimum. This implies we need to choose between $f(2)$ and $f(3)$, and if we raise both to the 18 power, we see that $\frac{1}{3^2}<\frac{1}{2^3}$.

26. A. Let m be the length of BD, a be the length of DC, and θ be the angle CBD. By the Law of Sines, we have that $a=\frac{\sin\theta}{\frac{3}{5}}m\Rightarrow -5=\frac{dm}{dt}+\frac{\frac{20}{3}d\theta}{dt}$ and by the Law of Cosines $a^2=m^2+64-16m\cos\theta\Rightarrow -50=10\frac{dm}{dt}-\frac{\frac{64}{5}dm}{dt}+48\frac{d\theta}{dt}\Rightarrow -50=-\frac{\frac{14}{5}dm}{dt}+48\frac{d\theta}{dt}$ by plugging in $a=5$ and $m=5$ since it is the median. Solving this system gives $\frac{dm}{dt}=\frac{7}{5}$.

27. C. If we take derivatives and plug in, we will see that our function needs to satisfy: $4(a^2-5a-12)e^{ax}+3(b^2-5b-12)e^{bx}=0$ which is solved at the two roots of the quadratic $f(x)=x^2-5x-12$. The sum of these (distinct) roots is 5.

28. B. For a product of monic monomials, we know that by the product rule that the derivative is the sum of all terms that are in the product divided by each of the individual monomial terms. This is exactly the integrand in question, which makes the integral determined by $P(x)$ evaluated at 1 and 0. $P(1)-P(0)=-2021!+2022!=2021\cdot 2021!$. We now compute the power of 2 in 2021!, which is 2013.

29. B. We can apply the ratio test to determine the radius of convergence: $\lim_{n\to\infty}\left|\frac{F_{n+1}}{(n+1)^2+1}\frac{n^2+1}{F_n}\right|=\frac{1+\sqrt{5}}{2}$, which makes the radius of convergence $\frac{1}{\frac{1+\sqrt{5}}{2}}=\frac{\sqrt{5}-1}{2}$.

30\. A. Using up to the x^2 term in each power series representation gives $\lim\limits_{x\to 0}\frac{1-(1-8x^2)}{x-(x+\frac{4x^2}{5})} = -10$.