Mu

Limits and Derivatives

Test #131

Directions:

1. Fill out the top section of the Round 1 Google Form answer sheet and select **Mu- Limits and Derivatives** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

1. Evaluate $\lim_{x\to 0}(1+x)^{\cot x}$

A. 0 B. 1 C.e D. Does Not Exist E. NOTA

2. Compute $\lim_{x\to 1}\frac{\left(x^{2021}-1\right)^2}{(x-1)^2}$

A. 2021 B. 2022 C.2021^2 D.2022^2 E. NOTA

3. Evaluate $\lim_{x\to 0}\frac{\arctan x-x}{\sin x-x}$

A. 0 B. $\frac{2}{3}$ C. $\frac{4}{3}$ D. 2 E. NOTA

4. If $f(x)=2g(2x)$, then compute the value of $\lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)}$

A. 1 B. 2 C. 4 D. 16 E. NOTA

5. Compute $\lim_{x\to 0}\frac{1-\int_0^x \sin t^2 dt}{x^3}$

A. -1 B. $-\frac{1}{3}$ C. $\frac{1}{3}$ D. Does Not Exist E. NOTA

6. Consider a function $g(f)$ that returns the greatest real root of a function $f(x)$, and 0 if $f(x)$ has no real roots. If $\lim_{a\to\infty} \frac{g(x^3-4ax+1)}{a^p} = k$ for some nonzero real number k and real number p, what is the value of pk?

A. 0 B. 1 C. 2 D. 4 E. NOTA

7. Let $A(n)$ be the area under the curve $f(x) = x^2 \sin x$ on $[0, \pi]$ as approximated by the trapezoidal rule using n equal width sub-intervals. Compute the value of $\lim_{n\to\infty} A(n)$

A. $\frac{\pi^2}{2} - 2$ B. $\pi^2 - 4$ C. $\frac{\pi^2}{2} + 2$ D. $\pi^2 - 2$ E. NOTA

8. If $f_n(x) = nx^n$ is a sequence of functions (indexed by n, a natural number), defined on [0,1], then compute the value of $\lim_{n\to\infty} \int_0^1 f_n(x)dx$

A. 0 B. 1 C. ∞ D. Does Not Exist E. NOTA

9. $f(x)$ is monotone increasing if $f'(x) \geq 0$ for all x in the domain. What is the minimum value of C that makes $f(x) = Cx - \frac{e^x}{e^x+1}$ monotone increasing on the reals?

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 1 D. 2 E. NOTA

10. $f(x)$ is convex if $f''(x) \geq 0$ for all x in the domain. What is the minimum value of C that makes $f(x) = 3\sin 2x + 4\cos 2x + Cx^2$ convex on $[0,2\pi)$?

A. -10 B. -5 C. 5 D. 10 E. NOTA

11. Which of the following describes $f(x) = \frac{\ln x}{x}$ at the point where $x = e$?

A. Local Minimum B. Local Maximum

C. Inflection Point D. Real Root E. NOTA

12. A function $f(x)$ has 3rd degree Taylor Polynomial $P(x) = 1 + x^2 + \frac{x^3}{2}$ around $x = 0$. Which of the following describes $f(x)$ at $x = 0$?

A. Local Minimum B. Local Maximum

C. Inflection Point D. Real Root E. NOTA

13. Compute $\frac{d}{dx}\int_0^1 2x^2y\,dy$, then evaluate at $x = 1$.

A. 0 B. 2 C. 4 D. 6 E. NOTA

14. If $f(x) = x^{1+x^2}$, then compute the value of $f'(1)$.

A. 2 B. 4 C. 6 D. 8 E. NOTA

15. If $\sin(x+y)=y$, evaluate $\frac{dy}{dx}$ at $(\pi,0)$.

A. -1 B. $-\frac{1}{2}$ C. $\frac{1}{2}$ D. 1 E. NOTA

16. If $\{x\}$ denotes the fractional part of a real number x and $[x]$ denotes the integer part of x, evaluate $\frac{d}{dx}(\{x\}^2[x]^2)$ at $x=2.022$.

A. 0.022 B. 0.044 C. 0.088 D. 0.176 E. NOTA

17. Let $f(x)=x^3+5$. The Mean Value Theorem for Derivatives on $f(x)$ and the interval $[a,b]$ guarantees a point $c\in(a,b)$ such that $f'(c)=13$. Which of the following is a possible ordered pair (a,b)? You may not consider subintervals within (a,b) to apply the MVT on.

A. (1,2) B. (2,3) C. (1,3) D. (2,4) E. NOTA

18. For how many integral values of k will the function $f(x)=x^3-kx^2+x-3$ have no turning points?

A. 0 B. 1 C. 2 D. 3 E. NOTA

19. Using the first order Taylor approximation of $f(x)=x^3-3x^2+4$ at $x=2$, what is the approximate value of $f(2.022)$?

A. 0 B. 0.044 C. 0.088 D. 0.352 E. NOTA

20. Using one iteration of Newton's Method and initial point $x_0 = 2$, the approximate solution to $f(x) = 0$ for $f(x) = x^3 - 5x + 1$ is $x = \frac{m}{n}$ for positive integers m and n. What is the minimum possible value of $m + n$?

A. 19 B. 20 C. 21 D. 22 E. NOTA

21. The tangent line to the curve $x^2 + y^2 = 1$ at $(\frac{3}{5}, \frac{4}{5})$ intersects the ellipse $25x^2 + y^2 = 25$ at a point in the first quadrant (m, n). Compute the value of $3m + 4n$.

A. 3 B. 4 C. 5 D. 6 E. NOTA

22. The graphs of $y = x^2 + ax + b$ and $y = x^3 + 4x + 1$ for a and b real numbers are tangent at $x = 1$. Compute the total area of region(s) bound by the two graphs.

A. $\frac{4}{3}$ B. 2 C. $\frac{8}{3}$ D. 4 E. NOTA

23. If $\sin^5 x + \cos^5 x = -1$, what is the minimum possible value of $\sin 5x$?

A. -1 B. $-\frac{1}{5}$ C. $-\frac{1}{25}$ D. 0 E. NOTA

24. A probability density function $p(x)$ over $[0,1]$ is given by $p(x) = Cx(1 - x)^4$, for C real. The mode of a density function $q(x)$ is the x that maximizes $q(x)$. What is the mode of $p(x)$?

A. $\frac{1}{5}$ B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$ E. NOTA

25. Which of the following real numbers is the smallest?

A. $\left(\frac{1}{5}\right)^{\frac{1}{15}}$ B. $\left(\frac{1}{4}\right)^{\frac{1}{12}}$ C. $\left(\frac{1}{3}\right)^{\frac{1}{9}}$ D. $\left(\frac{1}{2}\right)^{\frac{1}{6}}$ E. NOTA

26. Let triangle ABC be a right triangle with legs AB = 6, BC = 8, and right angle at B. Point D starts at vertex A at time t = 0 and moves towards C along hypotenuse AC at a rate of 5 units per seconD. At time t = 1 second, how fast, in units per second, is the length of BD increasing?

A. $\frac{7}{5}$ B. $\frac{28}{15}$ C. $\frac{21}{10}$ D. $\frac{14}{5}$ E. NOTA

27. Let $y = f(x) = 4e^{ax} + 3e^{bx}$ for a, b real numbers and $a \neq b$ be a solution to the second order ODE $y'' - 5y' - 12y = 0$. What is the maximum possible value of $a + b$?

A. -7 B. -1 C. 5 D. 11 E. NOTA

28. Let $P(x) = \prod_{i=2}^{2022}(x - i)$. The value of $\int_0^1 \sum_{i=2}^{2022} \frac{P(x)}{x-i} dx$ can be written as $m \cdot 2^n$ for integers m and n. What is the maximum value of n?

A. 2012 B. 2013 C. 2014 D. 2015 E. NOTA

For questions 29-30, let $f(x)$ be a function with power series representation

$$f(x) = \sum_{n=0}^{\infty} \frac{F_n}{n^2+1} x^n$$

Where $\{F_n\}$ is the Fibonacci Sequence with $F_0 = 0, F_1 = 1,$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

29. What is the radius of convergence of the power series?

A. $\frac{3-\sqrt{5}}{2}$ B. $\frac{\sqrt{5}-1}{2}$ C. $\frac{\sqrt{5}+1}{2}$ D. $\frac{3+\sqrt{5}}{2}$ E. NOTA

30. Compute the value of $\lim_{x \to 0} \frac{1-\cos 4x}{x-f(2x)}$.

A. -10 B. -8 C. -5 D. -4 E. NOTA