2015 – 2016 Log1 Contest Round 2
Theta Number Theory

Name: ________________

	4 points each	
1	Find the number of positive integral divisors of the number 4.	
2	Find the sum of the integers from 1 to 63, inclusive.	
3	How many of the following five integers are divisible by 7?: 6, 42, 245, 3122, 43659	
4	What fraction of the 50 least positive perfect squares are also perfect cubes?	
5	What fraction of the 50 least positive perfect cubes are also perfect squares?	

	5 points each	
6	Find the sum of the positive integral divisors of the number 4.	
7	What is the greatest common factor of 3458 and 3850?	
8	What is the least common multiple of 3458 and 3850?	
9	Find the two-digit number at the end of 1237^3 when it is expanded.	
10	How many positive integral factors of 2016 are divisible by 3?	

	6 points each	
11	When expanded, in how many consecutive zeros does $23!+24!$ end?	
12	In Java programming, there exists a command on integers called Integer Remainder Division, symbolized by %. For example, $20\%1=0$ since 20 leaves a remainder of 0 when divided by 1; also, $8\%3=2$ since 8 leaves a remainder of 2 when divided by 3. What would be the output on the Java command $3871183\%1967$?	
13	How many ordered pairs (x, y) of positive integers satisfy the equation $19x+29y=12673$?	
14	The sum of two positive prime numbers is 99. Find the product of these two prime numbers.	
15	"Take a k-digit integer n, where $k \geq 2$. Remove the units digit from n, resulting in a $(k-1)$- digit integer, then subtract twice the removed units digit from this $(k-1)$-digit integer. If the result of this is divisible by m, then n is divisible by m also." Find the least integer $m>1$ for which this divisibility rule is true.	

2015 – 2016 Log1 Contest Round 2
Alpha Number Theory

Name: ________________

	4 points each	
1	Find the number of positive integral divisors of the number 72.	
2	Find the sum of the integers from 1 to 63, inclusive.	
3	How many of the following five integers are divisible by 7?: 6, 42, 245, 3122, 43659	
4	What fraction of the 30 least positive perfect squares are also perfect *n*th powers for some integer $n > 2$?	
5	Find the remainder when 7^{2016} is divided by 5.	

	5 points each	
6	*a* and *b* are positive digits that are not necessarily distinct such that $\frac{a}{b}$ has equivalent decimal value $0.a\overline{b}$. Find the ordered pair(s) (a,b) that satisfy this description.	
7	Find the sum of the positive integral divisors of the number 72.	
8	Find the two-digit number at the end of 1237^5 when it is expanded.	
9	How many positive integral factors of 2016 are divisible by 4 but not 3?	
10	When expanded, in how many consecutive zeros does $23!+24!$ end?	

	6 points each	
11	In Java programming, there exists a command on integers called Integer Remainder Division, symbolized by %. For example, $20\%1=0$ since 20 leaves a remainder of 0 when divided by 1; also, $8\%3=2$ since 8 leaves a remainder of 2 when divided by 3. What would be the output on the Java command $3871183\%1967$?	
12	How many ordered pairs (x,y) of positive integers satisfy the equation $19x+29y=12673$?	
13	The sum of two positive prime numbers is 99. Find the product of these two prime numbers.	
14	"Take a *k*-digit integer *n*, where $k \geq 2$. Remove the units digit from *n*, resulting in a $(k-1)$- digit integer, then subtract twice the removed units digit from this $(k-1)$-digit integer. If the result of this is divisible by *m*, then *n* is divisible by *m* also." For how many integers *m*, where $2 \leq m \leq 10$, is this divisibility rule true?	
15	The prime factorization of the number 4,128,729,839 consists solely of a single copy each of six two-digit prime numbers. Find the sum of these six prime numbers.	

2015 – 2016 Log1 Contest Round 2
Mu Number Theory

Name: _______________

	4 points each	
1	Find the number of positive integral divisors of the number 2016.	
2	Find the sum of the integers from 1 to 63, inclusive.	
3	How many of the following five integers are divisible by 7?: 6, 42, 245, 3122, 43659	
4	What fraction of the 30 least positive perfect squares are also perfect *n*th powers for some integer $n>2$?	
5	*a* and *b* are positive digits that are not necessarily distinct such that $\frac{a}{b}$ has equivalent decimal value $0.a\bar{b}$. Find the ordered pair(s) (a,b) that satisfy this description.	

	5 points each	
6	Find the sum of the positive integral divisors of the number 2016.	
7	Find the two-digit number at the end of 1237^{10} when it is expanded.	
8	How many positive integral factors of 2016 are divisible by 4 but not 3?	
9	There are two right triangles with integer side lengths whose hypotenuse has length 481 such that the side lengths are relatively prime. Find the greater of the two triangles' perimeters.	
10	When expanded, in how many consecutive zeros does $23!+24!+25!$ end?	

	6 points each	
11	In Java programming, there exists a command on integers called Integer Remainder Division, symbolized by %. For example, $20\%1=0$ since 20 leaves a remainder of 0 when divided by 1; also, $8\%3=2$ since 8 leaves a remainder of 2 when divided by 3. What would be the output on the Java command $3871183\%1967$?	
12	How many ordered pairs (x,y) of positive integers satisfy the equation $19x+29y=12673$?	
13	The sum of two positive prime numbers is 99. Find the product of these two prime numbers.	
14	"Take a *k*-digit integer *n*, where $k\geq 2$. Remove the units digit from *n*, resulting in a $(k-1)$- digit integer, then subtract twice the removed units digit from this $(k-1)$-digit integer. If the result of this is divisible by *m*, then *n* is divisible by *m* also." For how many integers *m*, where $2\leq m\leq 10$, is this divisibility rule true?	
15	How many positive integers less than 1848 are not relatively prime with 1848?	

2015 – 2016 Log1 Contest Round 2
Theta Number Theory

Name: ________________

4 points each		
1	Find the number of positive integral divisors of the number 4.	3
2	Find the sum of the integers from 1 to 63, inclusive.	2016
3	How many of the following five integers are divisible by 7?: 6, 42, 245, 3122, 43659	4
4	What fraction of the 50 least positive perfect squares are also perfect cubes?	$\frac{3}{50}$
5	What fraction of the 50 least positive perfect cubes are also perfect squares?	$\frac{7}{50}$

5 points each		
6	Find the sum of the positive integral divisors of the number 4.	7
7	What is the greatest common factor of 3458 and 3850?	14
8	What is the least common multiple of 3458 and 3850?	950950
9	Find the two-digit number at the end of 1237^3 when it is expanded.	53
10	How many positive integral factors of 2016 are divisible by 3?	24

6 points each		
11	When expanded, in how many consecutive zeros does $23!+24!$ end?	6
12	In Java programming, there exists a command on integers called Integer Remainder Division, symbolized by %. For example, $20\%1=0$ since 20 leaves a remainder of 0 when divided by 1; also, $8\%3=2$ since 8 leaves a remainder of 2 when divided by 3. What would be the output on the Java command $3871183\%1967$?	127
13	How many ordered pairs (x,y) of positive integers satisfy the equation $19x+29y=12673$?	22
14	The sum of two positive prime numbers is 99. Find the product of these two prime numbers.	194
15	"Take a k-digit integer n, where $k\geq 2$. Remove the units digit from n, resulting in a $(k-1)$-digit integer, then subtract twice the removed units digit from this $(k-1)$-digit integer. If the result of this is divisible by m, then n is divisible by m also." Find the least integer $m>1$ for which this divisibility rule is true.	3

2015 – 2016 Log1 Contest Round 2
Alpha Number Theory

Name: ________________

4 points each		
1	Find the number of positive integral divisors of the number 72.	12
2	Find the sum of the integers from 1 to 63, inclusive.	2016
3	How many of the following five integers are divisible by 7?: 6, 42, 245, 3122, 43659	4
4	What fraction of the 30 least positive perfect squares are also perfect *n*th powers for some integer $n>2$?	$\frac{7}{30}$
5	Find the remainder when 7^{2016} is divided by 5.	1

5 points each		
6	*a* and *b* are positive digits that are not necessarily distinct such that $\frac{a}{b}$ has equivalent decimal value $0.a\overline{b}$. Find the ordered pair(s) (a,b) that satisfy this description.	$(1,6)$ and $(9,9)$
7	Find the sum of the positive integral divisors of the number 72.	195
8	Find the two-digit number at the end of 1237^5 when it is expanded.	57
9	How many positive integral factors of 2016 are divisible by 4 but not 3?	8
10	When expanded, in how many consecutive zeros does $23!+24!$ end?	6

6 points each		
11	In Java programming, there exists a command on integers called Integer Remainder Division, symbolized by %. For example, $20\%1=0$ since 20 leaves a remainder of 0 when divided by 1; also, $8\%3=2$ since 8 leaves a remainder of 2 when divided by 3. What would be the output on the Java command $3871183\%1967$?	127
12	How many ordered pairs (x,y) of positive integers satisfy the equation $19x+29y=12673$?	22
13	The sum of two positive prime numbers is 99. Find the product of these two prime numbers.	194
14	"Take a *k*-digit integer *n*, where $k\geq 2$. Remove the units digit from *n*, resulting in a $(k-1)$- digit integer, then subtract twice the removed units digit from this $(k-1)$-digit integer. If the result of this is divisible by *m*, then *n* is divisible by *m* also." For how many integers *m*, where $2\leq m\leq 10$, is this divisibility rule true?	2
15	The prime factorization of the number 4,128,729,839 consists solely of a single copy each of six two-digit prime numbers. Find the sum of these six prime numbers.	288

2015 – 2016 Log1 Contest Round 2
Mu Number Theory

Name: ________________

4 points each		
1	Find the number of positive integral divisors of the number 2016.	36
2	Find the sum of the integers from 1 to 63, inclusive.	2016
3	How many of the following five integers are divisible by 7?: 6, 42, 245, 3122, 43659	4
4	What fraction of the 30 least positive perfect squares are also perfect *n*th powers for some integer $n > 2$?	$\frac{7}{30}$
5	*a* and *b* are positive digits that are not necessarily distinct such that $\frac{a}{b}$ has equivalent decimal value $0.a\overline{b}$. Find the ordered pair(s) (a,b) that satisfy this description.	$(1,6)$ and $(9,9)$

5 points each		
6	Find the sum of the positive integral divisors of the number 2016.	6552
7	Find the two-digit number at the end of 1237^{10} when it is expanded.	49
8	How many positive integral factors of 2016 are divisible by 4 but not 3?	8
9	There are two right triangles with integer side lengths whose hypotenuse has length 481 such that the side lengths are relatively prime. Find the greater of the two triangles' perimeters.	1160
10	When expanded, in how many consecutive zeros does $23!+24!+25!$ end?	8

6 points each		
11	In Java programming, there exists a command on integers called Integer Remainder Division, symbolized by %. For example, $20\%1=0$ since 20 leaves a remainder of 0 when divided by 1; also, $8\%3=2$ since 8 leaves a remainder of 2 when divided by 3. What would be the output on the Java command $3871183\%1967$?	127
12	How many ordered pairs (x,y) of positive integers satisfy the equation $19x+29y=12673$?	22
13	The sum of two positive prime numbers is 99. Find the product of these two prime numbers.	194
14	"Take a *k*-digit integer *n*, where $k \geq 2$. Remove the units digit from *n*, resulting in a $(k-1)$- digit integer, then subtract twice the removed units digit from this $(k-1)$-digit integer. If the result of this is divisible by *m*, then *n* is divisible by *m* also." For how many integers *m*, where $2 \leq m \leq 10$, is this divisibility rule true?	2
15	How many positive integers less than 1848 are not relatively prime with 1848?	1367

2015 – 2016 Log1 Contest Round 2
Number Theory Solutions

Mu	Al	Th	Solution
		1	Since the positive integral divisors of 4 are 1, 2, and 4, there are 3 total.
	1		Since $72 = 2^3 \cdot 3^2$, each factor of 72 must contain any numbers of 2s, including none of them (4 options total) and any number of 3s, including none of them (3 options total). This gives $4 \cdot 3 = 12$ total positive integral factors of 72.
1			Since $2016 = 2^5 \cdot 3^2 \cdot 7$, each factor of 2016 must contain any numbers of 2s, including none of them (6 options total); any number of 3s, including none of them (3 options total); and any number of 7s, including none of them (2 options total). This gives $6 \cdot 3 \cdot 2 = 36$ total positive integral factors of 2016.
2	2	2	$\frac{63 \cdot (63+1)}{2} = 2016$
3	3	3	6 is not divisible by 7, but $42 = 7 \cdot 6$, $245 = 7 \cdot 35$, $3122 = 7 \cdot 446$, and $43659 = 7 \cdot 6237$, so 4 of the choices are divisible by 7.
		4	For a perfect square to also be a perfect cube, the number must be a perfect sixth power, meaning that the perfect square must be the square of a perfect cube. Since we are looking at the least 50 positive perfect squares, the numbers that are also perfect cubes are 1^2, 8^2, and 27^2, meaning that $\frac{3}{50}$ of the numbers fit the description.
4	4		1 will work as 1 to any number power is 1. For other perfect squares, any that are also perfect cubes would have to be perfect sixth powers, which would be 64 and 729. Any perfect fourth powers would be counted, which would be 16, 81, 256, and 625. Any fifth powers would have to be perfect tenth powers, which are too large—this applies also to seventh and ninth powers. Any sixth powers would be counted, but there aren't any that haven't already been counted. Any eighth powers would be counted, but there aren't any that haven't already been counted. Since tenth or larger powers would be too big for our set, we have a total of 7 numbers that are other powers, making the fraction $\frac{7}{30}$.
		5	For a perfect cube to also be a perfect square, the number must be a perfect sixth power, meaning that the perfect cube must be cube of a perfect square. Since we are looking at the least 50 positive perfect cubes, the numbers that are also perfect squares are 1^3, 4^3, 9^3, 16^3, 25^3, 36^3, and 49^3, meaning that $\frac{7}{50}$ of the numbers fit the description.
	5		In modulo 5, $7^{2016} = \left(7^4\right)^{504} = (2401)^{504} \equiv 1^{504} = 1$, so the remainder is 1.
		6	Since the positive integral divisors of 4 are 1, 2, and 4, their sum is $1+2+4=7$.

5	6		$\frac{a}{b}=\frac{a}{10}+\frac{b}{100}+\frac{b}{1000}+...=\frac{a}{10}+\frac{\frac{b}{100}}{1-\frac{1}{10}}=\frac{a}{10}+\frac{b}{100}\cdot\frac{10}{9}=\frac{a}{10}+\frac{b}{90}=\frac{9a+b}{90}\Rightarrow 90a=9ab+b^2$ $\Rightarrow a=\frac{b^2}{90-9b}=\frac{b^2}{9(10-b)}$, so b^2 must be a multiple of 9, or b must be a multiple of 3. $b=3$ yields $a=\frac{1}{7}$, which is not a solution. $b=6$ yields $a=1$, which is a solution ($\frac{1}{6}=0.1\overline{6}$). $b=9$ yields $a=9$, which is a solution ($\frac{9}{9}=0.9\overline{9}$, which is equivalent to 1). Therefore, the ordered pair solutions are $(1,6)$ and $(9,9)$.
6			Since $2016=2^5\cdot 3^2\cdot 7$, the sum of the positive integral divisors of 2016 is $(1+2+4+8+16+32)(1+3+9)(1+7)=(63)(13)(8)=6552$.
		7	Since the difference between 3850 and 3458 is $392=2^3\cdot 7^2$, we only need test 2s and 7s. Both 3850 and 3458 are divisible by 2 and 7 once each, so the greatest common factor of the two numbers is 14.
	7		Since $72=2^3\cdot 3^2$, the sum of the positive integral divisors of 72 is $(1+2+4+8)(1+3+9)=(15)(13)=195$.
7			At any point, we only need to look at the two-digit number at the end of any product—numbers in higher places than that will not affect the final two-digit number. Therefore, $37^2=1369$, $69^2=4761$, $61^2=3721$, and $69\cdot 21=1449$, so the final two-digit number of 1237^{10} is 49.
		8	Using the fact that the product of the greatest common factor and the least common multiple of two numbers equals the product of the two numbers, paired with the result of the last problem, the least common multiple of 3850 and 3458 is $\frac{3850\cdot 3458}{14}=950950$.
	8		At any point, we only need to look at the two-digit number at the end of any product—numbers in higher places than that will not affect the final two-digit number. Therefore, $37^2=1369$, $69^2=4761$, and $37\cdot 61=2257$, so the final two-digit number of 1237^5 is 57.
		9	At any point, we only need to look at the two-digit number at the end of any product—numbers in higher places than that will not affect the final two-digit number. Therefore, $37^2=1369$, and $69\cdot 37=2553$, so the final two-digit number of 1237^3 is 53.
8	9		$2016=2^5\cdot 3^2\cdot 7$, so to get factors that are divisible by 4 but not 3, they must contain at least two 2s (4 options total); any number of 7s, including none of them (2 options total); and no 3s (1 option total). This gives $4\cdot 2\cdot 1=8$ total factors divisible by 4 but not 3.

9			All primitive Pythagorean triples are of the form $\left(a^2-b^2,2ab,a^2+b^2\right)$, where *a* and *b* are integers. Thus we are trying to find integers *a* and *b* such that $a^2+b^2=481$. Without loss of generality, upon inspection, the only solutions are $(a,b)=(15,16)$ or $(a,b)=(9,20)$. This makes the two Pythagorean triples $(31,480,481)$ and $(319,360,481)$. The second one has the greater perimeter, which is $319+360+481$ $=1160$.
		10	$2016=2^5\cdot3^2\cdot7$, so to get factors that are divisible by 3, they must contain any numbers of 2s, including none of them (6 options total); any number of 7s, including none of them (2 options total); and one or two 3s (2 options total—there cannot be no 3s). This gives $6\cdot2\cdot2=24$ total factors divisible by 3.
	10	11	$23!+24!=(1+24)23!=25\cdot23!$, and the number of 0s in which this ends is dictated by the number of factors of 5 (there are fewer of them than factors of 2). Since $25\cdot23!=\frac{25!}{24}$, and 24 doesn't contain any factors of 5, this is the same number of 0s at the end of $25!$, which is $\left\lfloor\frac{25}{5}\right\rfloor+\left\lfloor\frac{25}{5^2}\right\rfloor+\left\lfloor\frac{25}{5^3}\right\rfloor+\left\lfloor\frac{25}{5^4}\right\rfloor+...=5+1+0+0+....=6$.
10			$23!+24!+25!=(1+24+25\cdot24)23!=25^2\cdot23!$, and the numbers of 0s in which this ends is dictated by the the number of factors of either 2 or 5 (whichever is fewer). 25^2 provides no additional factors of 2, so the number of factors of 2 in this number is $\left\lfloor\frac{23}{2}\right\rfloor+\left\lfloor\frac{23}{2^2}\right\rfloor+\left\lfloor\frac{23}{2^3}\right\rfloor+\left\lfloor\frac{23}{2^4}\right\rfloor+\left\lfloor\frac{23}{2^5}\right\rfloor+\left\lfloor\frac{23}{2^6}\right\rfloor+...=11+5+2+1+0+0+....=19$. The number of factors of 5 in $23!$ is $\left\lfloor\frac{23}{5}\right\rfloor+\left\lfloor\frac{23}{5^2}\right\rfloor+\left\lfloor\frac{23}{5^3}\right\rfloor+...=4+0+0+....=4$, and since 25^2 provides an additional four factors of 5, there are 8 total factors of 5. Therefore, this number ends in 8 consecutive zeros.
11	11	12	Using the division algorithm, $3871183=1967\cdot1968+127$, so the remainder (which is all this command is asking you to find) is 127.
12	12	13	One solution in integers to this equation is $(667,0)$ (although it isn't one we want to count). Now, since 19 and 29 are relatively prime, the other solutions may be found by subtracting (or adding) 29 from the *x*-coordinate of this solution and adding (or subtracting) 19 from the *y*-coordinate of the same solution (we only need to consider subtracting 29/adding 19, as subtracting 19 would yield negative integers for *y*). Doing this yields positive integers for the solutions for *y*-values from $19=19\cdot1$ up to $418=19\cdot22$. Once the *y*-value reaches $437=19\cdot23$, the *x*-value reaches 0 (another solution is $(0,437)$), so there are 22 such ordered pair solutions.
13	13	14	This problem seems to give too little information to solve, but considering that the sum of the two primes is 99, one prime must be odd and the other even. The only even prime is 2, making the two primes 2 and 97, which have a product of $2\cdot97=194$.

14	14		Obviously this divisibility rule works for 1, and it is the divisibility rule for 7, but for the other integers in the range, we must either verify the rule OR provide a counterexample. To verify this is not a divisibility rule for a certain number, we need find a number not divisible by *m* whose result by using this algorithm is divisible by *m*. A counterexample for $m=2$ is 41 (41 is not divisible by 2, but $4-2\cdot 1=2$ is divisible by 2). A counterexample for $m=4$ is 61 (61 is not divisible by 4, but $6-2\cdot 1=4$ is divisible by 4). A counterexample for $m=5$ is 71 (71 is not divisible by 5, but $7-2\cdot 1=5$ is divisible by 5). A counterexample for $m=6$ is 81 (81 is not divisible by 6, but $8-2\cdot 1=6$ is divisible by 6). A counterexample for $m=8$ is 101 (101 is not divisible by 8, but $10-2\cdot 1=8$ is divisible by 8). A counterexample for $m=9$ is 111 (111 is not divisible by 9, but $11-2\cdot 1=9$ is divisible by 9). A counterexample for $m=10$ is 121 (121 is not divisible by 10, but $12-2\cdot 1=10$ is divisible by 10). Remember the divisibility rule for 3: if the sum of the digits is divisible by 3, then the number itself is divisible by 3. Consider the *k*-digit number $a_{k-1}a_{k-2}...a_1a_0$, which is equivalent to $10^{k-1}a_{k-1}+10^{k-2}a_{k-2}+...+10^1a_1+a_0$. Removing the units' digit from the number and subtracting twice that digit from the remaining number yields $10^{k-2}a_{k-1}+10^{k-3}a_{k-2}+...+a_1-2a_0$. The sum of the digits of this new result is $a_{k-1}+a_{k-2}+...+a_1-2a_0$, which differs from the sum of the digits of the original number by $3a_0$ —therefore, if one of the numbers is divisible by 3, the other must be as well. That means this works as a divisibility rule for 3 also, making this a divisibility rule for 1, 3, and 7, or 3 numbers total.
		15	This is the divisibility rule for 7, but we must check to see if it works for a number less than that. To verify this is not a divisibility rule for a certain number, we need find a number not divisible by *m* whose result by using this algorithm is divisible by *m*. A counterexample for $m=2$ is 41 (41 is not divisible by 2, but $4-2\cdot 1=2$ is divisible by 2). Remember the divisibility rule for 3: if the sum of the digits is divisible by 3, then the number itself is divisible by 3. Consider the *k*-digit number $a_{k-1}a_{k-2}...a_1a_0$, which is equivalent to $10^{k-1}a_{k-1}+10^{k-2}a_{k-2}+...+10^1a_1+a_0$. Removing the units' digit from the number and subtracting twice that digit from the remaining number yields $10^{k-2}a_{k-1}+10^{k-3}a_{k-2}+...+a_1-2a_0$. The sum of the digits of this new result is $a_{k-1}+a_{k-2}+...+a_1-2a_0$, which differs from the sum of the digits of the original number by $3a_0$ —therefore, if one of the numbers is divisible by 3, the other must be as well. That means this works as a divisibility rule for 3 also.
	15		$4{,}128{,}729{,}839=13\cdot 23\cdot 43\cdot 53\cdot 73\cdot 83$, so the sum of these prime factors is $13+23+43+53+73+83=288$.
15			Since $1848=2^3\cdot 3\cdot 7\cdot 11$, the number of positive integers less than 1848 that are relatively prime with 1848 is $\phi(1848)=1848\left(\frac{10}{11}\right)\left(\frac{6}{7}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)=480$. This means that of the 1847 positive integers less than 1848, there are $1847-480=1367$ integers that are not relatively prime with 1848 (this number can also be found using an inclusion-exclusion argument).