

Alpha Equations and Inequalities Solutions
2007 Mu Alpha Theta National Convention

1. We can take the absolute values signs away from $4\sin^2(x)$ because it can't be negative. So we have nowhere where $\sin(x)$ can be negative, so we can take the positive square root of both sides. $2\sin(x) > 1$, so $\sin(x) > \frac{1}{2}$, so x is between $\pi/6$ and $5\pi/6$.

Answer: **B**

2. The first three functions' inverses are themselves, but the inverse of the fourth function is log to the base 2 of x .

Answer: **D**

3. For such values of t_0 , $x_1 = 1$, the particle x_2 never gets up to $x = 1$, and the particle x_3 passes $x = 1$ at time $t = e - 1 < \pi$.

Answer: **A**

4. $a^{-3} = -3 \Rightarrow a = (-3)^{-\frac{1}{3}}$

Answer: **D**

5. The sum of a complex number and its conjugate is twice the real part; therefore, the real part is $\frac{5}{2}$. To solve for the imaginary coefficient, we get $\sqrt{8 - \left(\frac{5}{2}\right)^2} = \frac{\sqrt{7}}{2}$, which is the absolute value of the imaginary part.

Answer: **E**

6. If $a = 64$, then $b = +8$ or -8 . If we choose $b = -8$, then $c = -2$ but we are left with no real solution for d . If we choose $b = +8$, then we have $c = 2$ and have two solutions for d , positive and negative. Therefore, there are two solutions for d .

Answer: **B**

7. $x^3 + 3x^2 + 3x + 1 = (x + 1)^3$, so all three solutions are $x = -1 \in \mathbf{R}$.

Answer: **C**

8. The arctangent function is always defined, whatever is in the square root sign must be positive or zero, so the logarithm to the base 2 must be negative or zero, so x^2 must be between 0 and 1, so the absolute value of x must be between 0 and 1. *x can be negative!*

Answer: **B**

9. $\sqrt[3]{\ln(\sin(2x))} = 0 \Rightarrow \ln(\sin(2x)) = 0 \Rightarrow \sin(2x) = 1 \Rightarrow \cos(2x) = 0 \Rightarrow \cos^2(x) - \sin^2(x) = 0$
 $\Rightarrow \cos^4(x) + \sin^4(x) - 2\cos^2(x)\sin^2(x) = (\cos^2(x) - \sin^2(x))^2 = 0$

Answer: **A**

10. Adding the two inequalities gives $s + t < r + p$. The negation of that statement is $s + t \geq r + p$.

Answer: **A**

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11. We do a change of bases on the left-hand side of the equation to make it

$$\log_b 2 - \frac{\log_b 2}{\log_b 2b} = \log_b 2 - \frac{\log_b 2}{\log_b 2 + \log_b b}. \text{ Now make the substitution } x = \log_b 2. \text{ So } b = 2^{1/x},$$

and the fact that b is greater than 1 implies that x is positive. So we have $x - \frac{x}{x+1} = \frac{1}{6}$, so,

multiplying both sides by $6 * (x + 1)$ gives us $6(x^2 + x - x) = x + 1$, which is equivalent to

$$6x^2 - x - 1 = 0. \text{ Solving this quadratic and taking the positive solution gives us } x = \frac{1}{2}.$$

Therefore, $b = 2^2 = 4$.

Answer: **B**

12. The lengths a and b cannot be equal, because then they would have to be the two legs of the triangle, and the hypotenuse would be an integer times the square root of two, which is irrational. Neither of them can be 7, because if one were 7, the other would be 6 or 8, and the third side of the right triangle would have to be an irrational number. Therefore, $c = 7$, a^2 and b^2 are consecutive perfect squares whose difference is $49 = 2 \cdot 24 + 1$, so the smaller of a and b is 24, and the larger is 25. $7 + 24 + 25 = 56$.

Answer: **C**

13. This polar coordinates graph maps out a circle centered at $(0, .5)$ with radius $1 / 2$.

Therefore, its area is $\pi (1/2)^2 = \pi/4$.

Answer: **A**

14. Statement I: Counterexample: $\frac{1}{2}x^2 + \frac{3}{2}x + 1$; FALSE.

Statement II: Counterexample: $x^2 - 5x + 4$; FALSE.

Statement III: If a is positive, the curve is concave up, and must first cross the x -axis going down and then going up, so the vertex is below the x -axis.

Answer: **A**

15. For choice c , this is not the correct half-angle formula; it is missing a \pm sign.

Answer: **C**

16. The facts are that n is divisible by 9 and that the remainder when n is divided by 223 is 2. Note that 2007 is the least common multiple of 223 and 9 (it is their product), so there is one unique possible remainder when n is divided by 2007. So, if $n = 2007c + x$, because $2007c$ is divisible by 9, then x itself is divisible by 9. Also, because $2007c$ is divisible by 223, then x itself must have a remainder of 2 when divided by 223. Choice B fits these criteria, because 225 is divisible by 9 and has a remainder of 2 when divided by 223.

Answer: **B**

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17. The sine function and the cosine function intersect at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, and it is between those two points that the sine is greater. Therefore, everywhere else, cosine is greater or equal.

Answer: **D**

18. The ratio $\frac{a_{n+1}}{a_n} = \frac{2007}{n+1}$, so when $n = 2006$, the ratio will be 1, and $a_{n+1} = a_n$. After that, the ratio will become less than 1, so the first time that $a_{n+1} > a_n$ is when $n = 2007$.

Answer: **D**

19. Statement I: By inspection, we can see that it has the trivial solution $x = y = z = 0$; IMPOSSIBLE.

Statement II: This is possible since there are three equations and three unknowns (and the determinant of the coefficient matrix is not necessarily zero); POSSIBLE

Statement III: This is *never* possible with a set of linear equations! IMPOSSIBLE

Statement IV: There will be an infinite number of solutions if the determinant of the coefficient matrix is zero; POSSIBLE

Answer: **B**

20. There are nine ninth roots of 1 in the unit circle on the complex number plane, which are the nine roots of the polynomial $a(x^9 - 1)$ for any constant a . All of these nine roots except for $\text{cis}(2\pi) = 1$ are roots of $p(x)$. So if we divide $a(x^9 - 1)$ by $(x - 1)$, giving us $a(x^8 + x^7 + \dots + 1)$, we have the general form of an eighth-degree polynomial with the roots described in the question. Since $p(0) = 1$, that makes $a = 1$, so $p(x) = x^8 + x^7 + \dots + 1$, so $p(1) = 9$.

Answer: **C**

21. Adding the first and third equation and then dividing by 2 gives us $z = 6$, so plugging into the first equation yields $x + y = 4$, so dividing the second equation by that tells us that $x - y = 2$, so $x = 3$ and $y = 1$. By inspection of the t^2 -coefficient and the constant coefficient, we can eliminate all choices except A, and we can check that it does indeed have those three solutions.

Answer: **A**

22. These implicit formulas are equivalent to $\frac{(x-4)^2}{4} + (y+1)^2 = 1$ and $(x-4)^2 + \frac{(y+1)^2}{4} = 1$, forming two ellipses both centered at $(4, -1)$. The distance between the center and all focal points is $\sqrt{2^2 - 1^2} = \sqrt{3}$, so the area is $2 \cdot (\sqrt{3})^2 = 6$.

Answer: **D**

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23. Choice A is the 2×2 identity matrix, which by definition can be multiplied on either side to another matrix without changing it, choice B is simply the same matrix as A, choice C is the inverse of A, which by definition can be multiplied to A on either side to produce the identity matrix, and choice D is simply a linear combination of the inverse and the identity matrix; therefore, it commutes in multiplication as well.

Answer: **E**

24. The vertices of the triangle are $(0, 0)$, $(2, 1)$, and $(2, -2)$. The area can be calculated to be $2 \cdot \frac{1}{2} + 2 \cdot \frac{2}{2} = 3$.

Answer: **A**

25. The prism is 1 unit across (the range of x values), 2 units high (the range of the y values), and 1.5 units deep (the range of the z values). So the volume is the product of those dimensions, which is 3.

Answer: **C**

26. The product of all seven of its solutions is $-\left(\frac{-8}{2}\right) = 4$. It has seven solutions total, so if there are six non-real solutions, there is exactly one real solutions. By inspection, one of the solutions is 1. So 4 divided by 1 equals the product of the remaining six (non-real) solutions.

Answer: **B**

27. There are 36 possible outcomes. The number of ways for the event in choice A to occur is 1, for choice B, 15, for choice C, 21, and for choice D, 6. Therefore, choice C has the event with the greatest probability.

Answer: **C**

28. After t seconds, $\frac{\pi}{3}t$ cubic centimeters of the sphere is gone, and is the volume of the cone.

The sphere's volume is $\frac{4}{3}\pi$ times its radius cubed, and the volume of the cone is also $\frac{4}{3}\pi$ times its radius cubed (because its height is equal to four times its radius). When the volumes are equal, the radii will also be equal. The initial volume of the sphere is $\frac{64}{3}\pi$. So when half of that has melted, the two volumes are equal. Therefore, the time is 32 seconds.

Answer: **A**

29. The first three choices are all paralleled by the hyperbolic function properties, but the fourth one is a bit different: $\cosh^2(x) - \sinh^2(x) = 1$, which is why they're associated with hyperbolas. (For those not familiar with the hyperbolic functions, this can be verified using the definitions supplied in the question.)

Answer: **D**

30. The square of the absolute value of $\sin(x)$ is $\sin^2(x)$. Furthermore, the absolute value of $\sin(x)$ is always between 0 and 1, and therefore cannot possibly be less than its square.

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Answer: **D**

Answer Key:

1. B
2. D
3. A
4. D
5. E
6. B
7. C
8. B
9. A
10. A
11. B
12. C
13. A
14. A
15. C
16. B
17. D
18. D
19. B
20. C
21. A
22. D
23. E
24. A
25. C
26. B
27. C
28. A
29. D
30. D