Alpha Gemini Test Solutions
2007 Mu Alpha Theta National Convention

==

1. B
2. C
3. A
4. C
5. D
6. A
7. A
8. C
9. E
10. B
11. A
12. D
13. A
14. D
15. E
16. B
17. D
18. E
19. A
20. B
21. C
22. C
23. D
24. B
25. B
26. B
27. C
28. B
29. C
30. C

1. **B.** MDCCLXXVI, so L.

2. **C.** $A=\sqrt{s(s-s_1)(s-s_2)(s-s_3)}$, where s is the semiperimeter and s_k are the lengths of the three sides. Plugging in what we know, the perimeter is 3*a*, so we get $18a=\sqrt{(3a/2)(3a/2-a+3)(a/2)(3a/2-a-3)} \Rightarrow 36=\sqrt{3\left(a^2/4-9\right)} \Rightarrow a=\pm 42$, but only the plus root works. So the perimeter is 3*a* = 126.

3. **A.** 4*C(13,5)/C(52,5) There are C(13,5) ways to get a flush of one suit (and 4 suits) and C(52,5) ways to get 5 cards from a deck. The multiplication comes out to $4*13*12*11*10*9/(52*51*50*49*48)$ or $33/16660$

4. **C.** The possible domain of $\ln(x)$ is $x>0$. For $\ln(\ln(x))$ the only values of $\ln(x)$ that would work must be for $x>1$. Then for $\ln(\ln(\ln(x)))$, the only values of $\ln(\ln(x))$ that would work would be for $x>e$.

5. **D.** A quick solution to this problem would be to let *a* = 1 and *x* and *y* both equal 0, then you get 1/2 + 1/2 = 1. For a more thorough solution, $\left(1+a^{x-y}\right)^{-1}+\left(1+a^{y-x}\right)^{-1}=\dfrac{1}{1+a^{x-y}}+\dfrac{1}{1+a^{y-x}}$, which equals $\dfrac{1}{1+a^{x-y}}+\dfrac{1}{1+a^{y-x}}=\dfrac{1+a^{y-x}+1+a^{x-y}}{\left(1+a^{x-y}\right)\left(1+a^{y-x}\right)}=\dfrac{2+a^{y-x}+a^{x-y}}{2+a^{y-x}+a^{x-y}}=1$

6. **A.** The formula for a lemniscate is $r^2=a^2\cos 2\theta$, thus A is correct.

7. **A.** $\tan(a-b)=[\tan(a)-\tan(b)]/[1+\tan(a)\tan(b)]$. $\tan\left[\left(\cos^{-1}(3/5)\right)\right]=4/3$ and $\tan\left[\left(\sin^{-1}(5/13)\right)\right]=5/12$. Plugging into the formula and simplifying gives 33/56.

8. **C.** $f=k\dfrac{m_1m_2}{d^2}$. For this question we have $k\dfrac{(3m_1)(4m_2)}{(d/2)^2}=48\left(k\dfrac{m_1m_2}{d^2}\right)=48f$.

9. **E.** The matrix is singular, and thus it has no inverse.

10. **B.** If you draw the points out on a graph, you'll see that $5\sqrt{2}$ is the hypotenuse and 5 is one of the legs of a 45-45-90 triangle. Thus the distance between the two points is the other leg, 5.

11. **A.** $\sum_{n=1}^{\infty}(2n-1)^{-1}(2n+1)^{-1}$, or with partial sums $\sum_{n=1}^{\infty}\left[(1/2)/(2n-1)\right]-\left[(1/2)/(2n+1)\right]$. If you expand out the first several terms of this series, you see that all terms cancel except the first 1/2, which is the answer.

12. **D.** The two lines are 160 miles apart. They are moving toward each other at 20 + 40 = 60 mph. Thus in 160/60 hours, or 8/3 hour at 5:40 PM, they will collide. In 8/3 hr, the line north of his location will move 20*8/3=160/3 miles in his direction. This is less than 60 miles, the distance between him and the north cloud line at the beginning…by a difference of 20/3 mile. So the collision will occur 20/3 miles north of his current location.

13. **A.** See solution for question number 12.

14. **D.** Writing out the formula from the determinant gets $(a-x)(b-x)-ab=0$. Simplifying gives $ab-(a+b)x+x^2-ab=0$, or $x^2-(a+b)x=0$, or $x\left(x-(a+b)\right)=0$. So the solutions are $x=\{0,\ a+b\}$.

15. **E.** The simplest way to answer this question is to convert each number to base 10, do the multiplication, then convert back to base 9. 33_9=30 and 44_9=40. 30 times 40 is 1200. This converts back to $1*9^3+5*9^2+7*9+3$, or 1573_9.

16. **B.** If $b^2-4ac>0$, where *b* is the coefficient on the *xy* term and *a* and *c* are the coefficients on the x^2 and y^2 terms, respectively, then the conic section is a hyperbola…if less than 0 an ellipse, and if equal to 0 a parabola. Here we have 3^2-4*7*(-4)=65>0. Thus it is a hyperbola.

17. **D.** If you have 1 choice only, then you could only have 1 addition. If you have 2 choices, then you can have 1 of each, or a combination…3 additions. For 3 additions, you have the 3 choices, then 3 combinations of 2, then 1 combination of all 3…7 additions. Note the pattern is 2^n-1. Thus for 10 addition choices, the answer is 1023.

18. **E.** A quarter of a mile is 440 yards. Thus the two semicircle ends have to be a total of 440 minus 2*150 = 140 yards. So the radius of each semicircle has length 140/π. If you have 6 lanes, then you must add 5 yards to the radius for Lane 6. For one lap, he would then walk $2\left((70/\pi)+5\right)\pi+2(150)=440+10\pi$. For 4 laps, this would be 1760+40π. Someone walking 4 laps around Lane 1 would go 1760 yards, thus the extra amount walked is 40π.

19. **A.** The full expansion of the dot product gives $(x^2+3x)(x-3)+4x(x+3)=0$, or $x^3+4x^2+3x=0 \Rightarrow x(x+1)(x+3)=0$, so the solutions are $x=\{0,-1,-3\}$.

20. **B.** All the snail has to do is reach the top. After its climb on day 1, it has made it to 5 feet. On day 2, it has made it to 6 feet. Following the progression, on day 96 it makes it to 100 feet.

21. **C.** Let $y = \sin(2x)$, then the problem comes down to solving a quadratic equation. You get $2y^2 + y - 1 = 0 \Rightarrow (2y-1)(y+1) = 0$. Then replace *y* back in and solve for *x*. $\sin(2x) = 1/2$ and $\sin(2x) = -1$. The solutions to the first are $x = \{\pi/12, 5\pi/12, 13\pi/12, 17\pi/12\}$ in the given domain and to the second are $x = \{3\pi/4, 7\pi/4\}$. So there are 6 solutions.

22. **C.** The resultant of the two vectors is the sum $\langle -7, 6, -6 \rangle$, which has a magnitude of $\sqrt{(-7)^2 + 6^2 + (-6)^2} = \sqrt{121} = 11$. To make this vector a unit vector the magnitude must be 1. If you want it parallel to the resultant, then divide $\langle -7, 6, -6 \rangle$ by 11, to get $\langle -7/11, 6/11, -6/11 \rangle$.

23. **D.** Represent the base *i* as $e^{(\pi/2 + 2\pi k)i}$, where $|k|$ is a whole number. Then you have $\left(e^{(\pi/2+2\pi k)i}\right)^i = e^{(\pi/2+2\pi k)i^2} = e^{-(\pi/2+2\pi k)}$. For $k = 0$, the only correct answer choice is $e^{-\pi/2}$.

24. **B.** For the 2 2-digit squares, you have 6 ways to choose the first and then 5 to choose the second (because they have to be different square numbers), so there are 30 combinations here. Finally there are 68 numbers whose squares come out to 4 digits (32^2 through 99^2). This makes for a total of 98. Next you have to look for repeats! The easiest way to check for this is to write out the 30 combinations of 2 2-digit squares. The number 1681 is both 41^2 and 4^2+9^2. This turns out to be the only repeat, as you'll note that 50*50=2500 and then 51*51=2601, and the same is true for all of the higher 4-digit numbers. Thus the answer is 97.

25. **B.** If the roots are going to be rational, then the only possible roots are ± 1, $\pm p$, $\pm q$, and $\pm pq$. Now the sum of the roots must be -1, so the absolute difference between two of the roots must be 1. Only the prime numbers 2 and 3 fit this profile, as 1 is not a prime number. Now you must find out which roots work in the equation (go through all of the plus/minus combinations). You find that only the choice with *x* = {2 or -3} works, so there is only 1 set of such roots.

26. **B.** Average speed is harmonic mean of 50 & 70, or 2/[(1/50)+(1/70)]=175/3.

27. **C.** The equation breaks down to an ellipse with form: $\frac{(x-2)^2}{1^2} + \frac{(y-1)^2}{3^2} = 1$. The area of an ellipse is given by πab, where *a* is 3 in this case and *b* is 1. Thus the area is 3π.

28. **B.** For $y = \tan(Ax - B)$, the periodicity is π/A, so the answer is $\pi/8$.

29. **C.** $f[g(x)] = 6x^2 + 15x - 14 = 0$. Using quadratic equation gives $x = \frac{-15 \pm \sqrt{561}}{12}$.

30. **C.** The volume is $4x^2$, where x is the length of a side of the base. Thus $x = 2\sqrt{3}$. The surface area will then be the four sides plus the base times 2 (both the inside and the outside of the box has exposed surface area… $2\left(4\times 8\sqrt{3}+\left(2\sqrt{3}\right)^2\right) = 64\sqrt{3}+24$.