Matrices & Vectors Alpha solutions
2007 Mu Alpha Theta National Convention

1. A is a 2x2 matrix, B is a 3x3 matrix, and C is a 3x2 matrix. We can only multiply any two matrices X and Y, in that order, if the second dimension of X matches the first dimension of Y. Therefore, I and III are possible, whereas II and IV are not. The answer is D.
2. The trace, or the sum of the diagonal elements, 4+1+3 = 8, which is not listed as a choice. The answer is E.
3. These are the same matrices as in problem 2. We have already computed AB. BA = $\begin{bmatrix} 1.5 & 5 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 12 \\ -12 & 5 \end{bmatrix}$. So, AB – BA = $\begin{bmatrix} 11 & 12 \\ -6 & 17 \end{bmatrix} - \begin{bmatrix} 23 & 12 \\ -12 & 5 \end{bmatrix} = \begin{bmatrix} -12 & 0 \\ 6 & 12 \end{bmatrix}$. The answer is C.
4. A singular matrix is one that does not have an inverse, or equivalently, one that has a determinant of zero. Since Det(AB) = Det(A)Det(B), statements II and III must be true. If S is a singular matrix, Det(SS) = Det(S)Det(S) = 0*0 = 0. Also, Det(S*A) = Det(S)Det(A) = 0*Det(A) = 0. Statements I and IV can be shown to not be true. For example, the sum of the following two singular matrices is not singular: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. The answer is B.
5. To find the eigenvalues of D, we take $\begin{vmatrix} 1-\lambda & 7 \\ 42 & -6-\lambda \end{vmatrix}$ and set equal to zero and solve. The determinant leaves us with the quadratic $\lambda^2 + 5\lambda - 300$, which has the solutions λ = -20 and λ = 15. The larger of these two is 15, and the smaller is -20. Therefore, 3x + 2y = 45 – 40 = 5. The answer is A
6. M = $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, M^2 = $\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$, M^3 = $\begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix}$, … It can be shown by induction that M^n = $\begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, and the sum of these entries is $2(2^n) = 2^{n+1}$. The answer is B.
7. The inverse of matrix A is $\dfrac{\begin{bmatrix} 6 & 1 \\ -x & 4 \end{bmatrix}}{\det(A)} = \dfrac{\begin{bmatrix} 6 & 1 \\ -x & 4 \end{bmatrix}}{24+x}$. Given that this matrix has an entry of 1, the only values in the denominator that will keep this number an integer are ones that make 24+x equal to +/-1. Therefore, x can equal either -23 or -25, and the product of these two numbers is 575. The answer is A.
8. For there to be no solution to the system, the determinant of the coefficient matrix must be equal to zero. $\begin{vmatrix} 2 & 5 & 1 \\ -4 & b & 6 \\ 0 & -3 & -b \end{vmatrix}$ = $2(-b^2+18) - 5(4b) + 1(12) = -2b^2 + 36 - 20b + 12 = -2b^2 + 20b + 48 = 0$. The product of the roots of this equation is equal to 48/-2, or -24. The answer is C.
9. We know that if θ is the angle between the two vectors, then cos(θ) = $\dfrac{<4,1,8> \bullet <8,24,6>}{|<4,1,8>||<8,24,6>|} = \dfrac{104}{9*26} = \dfrac{4}{9}$. Since the cosine of the smaller angle (which must be between 0 and 180 degrees) is positive, we know that this angle must in fact be between 0 and 90 degrees, otherwise its cosine would be negative. Therefore, the angle also has a positive sine. Since $\sin^2(\theta) = 1-\cos^2(\theta)$, $\sin^2(\theta) = 65/81$, so $\sin(\theta) = \dfrac{\sqrt{65}}{9}$. The answer is D.

Matrices & Vectors Alpha solutions
2007 Mu Alpha Theta National Convention

10. (a x b) = $\begin{vmatrix} i & j & k \\ 0 & -4 & 5 \\ -2 & 3 & -2 \end{vmatrix}$ = -7i - 10j - 8k. The dot product of this vector, <-7,-10,-8>, with <-11,-10,1> is 77 + 100 – 8 = 169. Since the square root of 169 is 13, the answer is D.
11. A symmetric matrix is a matrix M that has the property that $M^T = M$. Only choice A satisfies this criterion. The answer is A.
12. The determinant is 2W - 4X, and the determinant is said to equal 10. if W+X = 8, then 4W + 4X = 32. Adding these two equations together shows that 6W = 42, or W = 7. Sine W+X=8, X =1. Therefore, WX = 7. The answer is B.
13. An orthogonal matrix B must satisfy $B^T = B^{-1}$. Therefore, the answer is B.
14. Det(AB) = Det(A)Det(B). Since Det(AB) = 170, and Det(A) = 4*5-1*3=17, it must be that Det(B) = 10. The only choice that has a determinant of 10 is C. The answer is C.
15. If we anchor ourselves at the point (-4,1,-2), we obtain the two vectors (0,2,1)-(-4,1,-2) = <4,1,3> and (-2,0,3)-(-4,1,-2) = <2,-1,5>. Then, the area of the triangle can be determined by taking the magnitude of the cross product of these two vectors, divided by two. This is equal to: $\frac{1}{2}\left\|\begin{matrix} i & j & k \\ 4 & 1 & 3 \\ 2 & -1 & 5 \end{matrix}\right\|$ $= \frac{1}{2}\left|< 8,14,-6 >\right| = \frac{1}{2}\sqrt{6^2+14^2+8^2} = \frac{\sqrt{296}}{2} = \sqrt{74}$. The answer is D.
16. All 4 choices have relatively easy determinants to take. Since choices A and D are triangular, their determinant is just the product of their diagonal numbers. Since choice B is 2x2, that determinant is also simple. Choice C has a row of zeroes, which automatically makes its determinant zero. The determinants of A, B, C, and D, respectively, are 100, 72, 0, and -105. Therefore, the answer is A.
17. The rank of a matrix is the number of linearly independent rows or columns it has. In choices A, B, and C, we can see that the first row is a multiple of the second row (in addition, the first column is a multiple of the second column). However, choice D has 2 linearly independent rows. Therefore, the rank of choice D is 2, and the answer is D.
18. The projection of a vector u onto a vector v is given by $\frac{u \bullet v}{|u||v|}u$. So, the magnitude of this vector is $\frac{u \bullet v}{|u||v|}|u| = \frac{u \bullet v}{|v|}$. In this case, we have $\frac{< 4,5,2 > \bullet < 12,-3,4 >}{\sqrt{12^2+(-3)^2+4^2}} = \frac{41}{13}$. The answer is C.
19. If a plane flies for 2 hours at x miles per hour at a bearing of 120 degrees, which is 30 degrees south of west, it will end up 2x units from the origin, at the point (-$\sqrt{3}$x,-x). If it then switches its bearing to 270 degrees, which is east, he is of course $\sqrt{3}$x units from the y-axis. Therefore, it will take him $\sqrt{3}$x / x = $\sqrt{3}$ hours. The answer is C.
20. Looking at M^2, M^2 =
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta)-\sin^2(\theta) & -2\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & \cos^2(\theta)-\sin^2(\theta) \end{bmatrix} =$$
$\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$. So, choice D looks the most attractive so far. In fact, we can prove just as easily that for any positive integers i and j,
$\begin{bmatrix} \cos(i\theta) & -\sin(i\theta) \\ \sin(i\theta) & \cos(i\theta) \end{bmatrix}\begin{bmatrix} \cos(j\theta) & -\sin(j\theta) \\ \sin(j\theta) & \cos(j\theta) \end{bmatrix} = \begin{bmatrix} \cos((i+j)\theta) & -\sin((i+j)\theta) \\ \sin((i+j)\theta) & \cos((i+j)\theta) \end{bmatrix}$. Therefore, if we

simply iterate this multiplication N times, we will indeed end up with $\begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$, which is choice D. The answer is D.

21. $<4,3,6> \bullet <x,2,y>$ = 4x + 6 + 6y = 18. Also, $<y,x> \bullet <3,4>$ = 3y + 4x = 18. Therefore, we have $\begin{array}{c} 4x + 6 + 6y = 18 \\ 4x + 3y = 18 \end{array} \Rightarrow \begin{array}{c} 4x + 6y = 12 \\ 4x + 3y = 18 \end{array}$. By subtracting the second equation from the first, we see that 3y = -6, so y = -2. Then we plug that in to find that x = 6. So, x – y = 6 – (-2) = 8. The answer is B.

22. To find the equation of the plane through these points, we first will "anchor" ourselves at (1,0,3). The two vectors determined by these points are (4,-2,5)-(1,0,3) = $<3,-2,2>$ and (7,7,1)-(1,0,3) = $<6,7,-2>$. The equation of the plane that contains these two vectors, if written as Ax+By+Cz+d = 0, will have as A, B, and C, the three components of the vector normal to the two this plane contains. We can figure this vector out by taking the cross product: $\begin{vmatrix} i & j & k \\ 3 & -2 & 2 \\ 6 & 7 & -2 \end{vmatrix} =$ $<-10,18,33>$.
So, the plane should be written -10x + 18y + 33z + D = 0. However, we want A > 0, so we multiply this by -1 to obtain the plane 10x – 18y – 33z + D = 0. To find D, we need only plug in one of the 3 points that this plane was supposed to contain: (1,0,3) looks like the easiest. 10(1) – 0 – 33(3) + D = 0 → D = 89. Therefore, the plane is 10x – 18y – 33z + 89 = 0. The sum of these three coefficients is 10-18-33+89 = 48. The answer is B.

23. If A = $\begin{bmatrix} 2 & -3 \\ 6 & -8 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & -1 \\ 3 & 6 \end{bmatrix}$, then 2B+A = $\begin{bmatrix} 8 & -2 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 6 & -8 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ 12 & 4 \end{bmatrix}$, the determinant of which equals 10*4 – (-5)*12 = 100. The answer is D.

24. $\begin{bmatrix} x+y & 8 \\ x & -2 \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ x \end{bmatrix} \Rightarrow \begin{array}{c} (x+y)(x) + 24 = 18 \\ x^2 - 6 = x \end{array}$. The second of these two equations can easily be factored into (x-3)(x+2) = 0, which means x can be either 3 or -2. If x is 3, then we have (3+y)(3) + 24 = 18, and so y = -5. If x is -2, then we have (-2+y)(-2) + 24 = 18, and so y = 5. Therefore, the sum of the possible values for y is 5-5 = 0. The answer is A.

25. When we convert the two vectors to rectangular coordinates, they become $<-2,2>$ and $<1,\sqrt{3}>$. The dot product of these two vectors is $2\sqrt{3} - 2$. The answer is C.

26. The total amount of money gained/lost by John and Fred combined can be represented several different ways, and choice A is one of them. An easy way to tell that the answer is not B, C, or Di s that none of these choices result in a 1x1 matrix. One way of calculating the desired number is:
$\begin{bmatrix} 25 & -70 & 50 & 110 \end{bmatrix}\begin{bmatrix} 40 & 100 \\ 90 & 30 \\ 150 & 60 \\ 75 & 80 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The answer is A.

27. The magnitude is $\sqrt{5^2 + (-6)^2 + 4^2} = \sqrt{77}$. The answer is C.

28. The rotation results in the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and reflecting this over y = x results in the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore the answer is B.

29. We need to convert the two lines to rectangular coordinates from parametric before we can determine if they intersect, and where. At t = 0, the first line has a point at (2,4), and at t = 1 it has a point at (5,3). Therefore, the first line is the line between these two points, which is the line

3y+x=14. At t=0, the second line has a point at (5,-2), and at t=1 it has a point at (9,0). Therefore, the second line is the line between these two points, which is the line 2y-x=-9. Adding these two lines together, we see that their point of intersection occurs when 5y = 5, or when y = 1. Plugging in, we see that at this value for y, x = 11. Therefore, the answer is B.

30. By multiplying the scalars by their respective vectors, and adding the vectors, we see that $\begin{bmatrix} x^2 + y^2 \\ 2xy \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$. Adding these two components together, we see that $x^2+y^2+2xy = 18$, which of course means that $(x+y)^2 = 18$. This means that the quantity x+y can either be positive or negative $3\sqrt{2}$. Therefore, the answer is A.