Mu Alpha Theta National Convention: Tampa Bay, 2007
Number Theory Topic Test Solutions – Alpha Division

1. **(B)**. We have $72 = (2^3)(3^2)$ and $210 = (2)(3)(5)(7)$ so their greatest common factor is $(2)(3) = 6$.

2. **(D)**. We have $56_7 = 5(7) + 6(1) = 41$ in base 10. Since $41 = 5(8) + 1(1)$, the answer is 51.

3. **(C)**. All arithmetic is done modulo 7. We have $5^5 \equiv (-2)^5 \equiv -32 \equiv 10$, which has a remainder of 3 upon division by 7.

4. **(C)**. Since $1555200 = (2^8)(3^5)(5^2)$, we have $(8 + 1)(5 + 1)(2 + 1) = 162$ positive integral factors.

5. **(D)**. By the divisibility rule for 11, if the alternating sum of the digits is divisible by 11, the original number is also divisible by 11. Since $9 - 4 + 1 - 0 + 2 - 5 + 4 - 2 + 6 = 11$, answer choice D is the answer. The other answer choices can be eliminated by applying the divisibility rule accordingly.

6. **(D)**. Any number divisible by 72 is also divisible by 4. Therefore, 6B must be a multiple of 4, making B either 0, 4, or 8. Any number divisible by 72 is also divisible by 9. By the divisibility test for 9, the sum of the digits must be a multiple of 9. Thus, A + B must leave a remainder of 6 upon division by 9. A little trial-and-error shows that A must be either 2 or 7, and their product is 14.

7. **(B)**. All integers congruent to 3 mod 8 have remainder 3 when divided by 8. We only need to worry about the last three digits of the number, as any piece greater than 1000 will be congruent to 0. Since 611 leaves a remainder of 3 upon division by 8, this is the answer.

8. **(A)**. In congruence notation, the problem reduces to $x \equiv 2 \bmod 3$, $x \equiv 3 \bmod 5$, and $x \equiv 2 \bmod 7$. From the first equation, we know that $x = 3a + 2$ for some integer a. Plug this into the third equation to get $a = 7b$ for some integer b, making $x = 21b + 2$. Plug this into the third equation and we get $b = 1 \bmod 5$, or $b = 5c + 1$ for some integer c, making $x = 21(5c + 1) + 2 = 105c + 23$. Therefore, the smallest positive integer solution to the system is $x = 105 + 23 = 128$. The product of its digits is 16.

9. **(B)**. Since $196 = (2^2)(7^2)$, $s(196) = ((2^3 - 1) / (2 - 1))((7^3 - 1) / (7 - 1)) = 399$.

10. **(D)**. From the factorization above, 196 has $(2 + 1)(2 + 1) = 9$ positive integral factors. Thus, by the standard formula, the product of these factors is $(196)^{9/2} = (14^2)^{9/2} = 14^9$.

11. **(A)**. A basic result is that $\gcd(m, n)\mathrm{lcm}(m, n) = mn$, so the answer is $(2007)(1001) = (2007)(1000 + 1) = 2007000 + 2007 = 2009007$.

12. **(C)**. If an integer has an odd number of positive integral factors, it's a perfect square. Thus, the problem amounts to finding the number of perfect squares in the given range. Since $100 = 10^2$ and $10000 = 100^2$, there are 100 – 9 – 2 = 89 integers with the desired property.

13. **(A)**. Since $196 = (2^2)(7^2)$, by the standard formula, $\phi(196) = 196(1 - 1/2)(1 - 1/7) = 84$.

14. **(E)**. Let the two numbers equal a and b, where a = 19b. If a is a two-digit number, then it must be one of {19, 38, 57, 76, 95}. However, the allowed digits to use forbids this possibility. If a is a three-digit number, then it must be one of {114, 133, 152, 171}. Again, given the digits we are allowed to use, this case also isn't possible. So x is impossible to compute.

15. **(B)**. From the standard formula, the answer is (7)(13) – 7 – 13 = 71.

16. **(B)**. The ten smallest Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, and 59. The sum of these is 143, which has a remainder of 2 upon division by 3. As an alternative to adding up these numbers, we could use the fact that the sum of the first n Fibonacci numbers is F(n + 2) – 1. Since in this problem n = 10, we have F(12) – 1 = 144 – 1 = 143, producing the same result.

17. **(C)** The first multiple of 5 in the terms of the sum is 5! = 120. Any term beyond that will immediately be divisible by 5 as well, so the problem is equivalent to finding the remainder when 1! + 2! + 3! = 9 is divided by 5, hence the answer is 4.

18. **(A)**. Given $P(x) = ax^2 + bx + c$, the discriminant is $b^2 - 4ac$. In modulo 4, this expression reduces to just b^2. If b is odd, then this expression is equivalent to 1 and if b is even, it is congruent to 0 mod 4. Thus, all the possible values of the discriminant is either congruent to 0 or 1 in modulo 4. All the answer choices satisfy this requirement except for answer choice A, which has a remainder of 3 upon division by 4.

19. **(C)**. Repeated exponentiation yields x = 5^(3^(2^(2007))) = $5^{\text{something}}$; the largest prime factor of this number is 5.

20. **(B)**. Let $F(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$. Now $7^a - 5^a$ in modulo 2 is 0, since it is the difference of two odd integers. Thus, F(7) – F(5) is an even integer, since it is just a linear combination of integers of the form $7^a - 5^a$. Similarly, in modulo 5 we have $7^a - 2^a \equiv 2^a - 2^a \equiv 0$, making F(7) – F(2) a multiple of 5. Using the given information, we conclude that F(7) is both a multiple of 2 and 5, hence, a multiple of 10.

21. **(B)**. Since $36 = 2^23^2$, we must have $x = 2^a3^b$ and $y = 2^c3^d$, where max(a, c) = max(b, d) = 2. These equations each have 5 solutions, taking into account the overlapping case when a = c = b = d = 2. Thus, the total number of solutions is (5)(5) = 25.

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22. **(B)**. We know that $y < 38$, since the sum of the divisors includes at least y and 1. Notice that 37 is prime, so that $\sigma(37) = 1 + 37 = 38$. Thus, $y = 37$, and the sum of the digits of y is $7 + 3 = 10$.

23. **(D)**. Any base-7 number can be written as $a_k\, 7^k + a_{k-1}\, 7^{k-1} + \ldots + a_0$. Taking this in modulo 6, we get $a_k + a_{k-1} + \ldots + a_0$. Thus, a base-7 number is divisible by 6 if the sum of its digits is divisible by 6. Answer Choice D qualifies for this since $2 + 5 + 3 + 4 + 3 + 5 + 2 = 24$, which is a multiple of 6.

24. **(A)**. 5 is prime, 1 is neither prime nor composite, 16 and 1000 are both composite, and of course, 23 is prime. There are two good values out of the five, or 40%.

25. **(C)**. There are no solutions to this equation, via Fermat's Last Theorem (a proof of which has no reasonable room for inclusion in this document!).

26. **(D)**. We have $3x \equiv 5 \bmod 7$, or $3x \equiv 12 \bmod 7$, or $x \equiv 4 \bmod 7$. The possible values of x are of the form $7x + 4$; the second smallest positive value is 11.

27. **(E)**. Anna's polynomial might look like $17x^{2007} + a_{2006}x^{2006} + \ldots + 41$. Notice that when 41 is plugged-in for x, all the terms will contain a 41, making the expression composite. Thus, that is the answer.

28. **(C)**. Both addends are odd, making the sum even.

29. **(D)**. Following the hint, we are motivated to assume that all desired integers are of the form $100k \pm 38$ for some integer k. We now show that this is the case. Squaring this quantity modulo 1000 yields $100^2k^2 + 7600k + 1444 \equiv 600k + 444 \equiv 100(6k + 4) + 44$. The number will have three 4's when $6k + 4$ has units digit of 4; this occurs when k is a multiple of 5, i.e. $k = 5t$ for some t. Thus, the numbers needed are of the form $100k \pm 38 = 500t \pm 38$. The smallest of these numbers is $500 - 38 = 462$, and the sum of the digits is $4 + 6 + 2 = 12$.

30. **(B)**. Let (x, y) be a lattice point that passes through the given line. The equation of the line is $y = mx$, or $m = y / x$. Since x and y are integers, m must be a rational number. The only rational