Alpha Probability Solutions
2007 Mu Alpha Theta National Convention

1. D
2. C
3. A
4. A
5. B
6. D
7. B
8. D
9. C
10. C
11. A
12. D
13. B
14. A
15. C
16. B
17. D
18. E
19. C
20. B
21. C
22. B
23. D
24. C
25. E
26. A
27. B
28. C
29. C
30. A

1. If A occurs with certainty, then the probability that exactly 2 of A, B, and C occur is the same as the probability that exactly 1 of B and C occur. Since P(B) = .5 and P(C) = .7, P(B and not C) = .5 (1-.7) = .15, and P(C and not B) = .7(1-.5) = .35. So, P(one of B or C) = .15 + .35 = .50, or ½.   D.

2. On the square with vertices (1,1), (1,-1), (-1,-1), and (-1,1), the areas where |X+Y| > .5 are the areas above the line y = -x + .5 and below the line y = -x - .5. Each of the aforementioned areas is equal to .5*(1.5)*(1.5) = 1.125, and therefore the total area for which |X+Y| > .5 = 1.125 + 1.125 = 2.25. Therefore, the probability that |X+Y| > .5 = 2.25/(area of square) = 2.25/4 = 9/16.   C.

3. Suppose there are n families. There must be n daughters. ½ of the time there will be 0 boys (i.e. a daughter is had first). ¼ of the time there will be 1 boy (i.e. daughter, then son). Proceeding in this fashion, we can see that the expected number of boys will be E(boys) $= n*[\ 0*\frac{1}{2}+1*\frac{1}{4}+2*\frac{1}{8}+3*\frac{1}{16}+4*\frac{1}{32}+...\ ]$

   So, we can see that 2*E(boys) $= n*[1*\frac{1}{2}+2*\frac{1}{4}+3*\frac{1}{8}+4*\frac{1}{16}+...]$, and that we

   can now say E(boys) $= [\ 2*E(boys) - E(boys)] = n*\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i} = n*1 = n.$

   Therefore, the ratio of boys to girls is expected to be n:n, or 1:1.   A.

4. There are 4 jacks, 4 queens, and 4 kings, so the probability is (4+4+4)/52 = 3/13.   A.

5. Suppose the game takes E(T) time to complete, where E(T) denotes expected time. Should we roll any number k in {1,2,3,4,5}, we know we will have to wait k minutes before repeating the game as if nothing has happened. Therefore, given we roll k in {1,2,3,4,5}, we would expect the game to take k + E(T) minutes. If we roll a 6, however, the game is over and this takes no extra time. Therefore, we can say:

   $$E(T) = \frac{1}{6}[E(T)+1]+\frac{1}{6}[E(T)+2]+\frac{1}{6}[E(T)+3]+\frac{1}{6}[E(T)+4]+\frac{1}{6}[E(T)+5]+\frac{1}{6}*0.$$
   Simplifying, we find that E(T) = 15.   B.

6. By Bayes Theorem, p(Ball 1 Red | Ball 2 Red) =

   $$\frac{p(Ball2red\ |\ Ball1red)*p(Ball1red)}{[p(Ball2red\ |\ Ball1red)*p(Ball1red)]+[p(Ball2red\ |\ Ball1green)*p(Ball1green)]}$$

   $$=\ \frac{\frac{1}{2}*\frac{5}{8}}{\frac{1}{2}*\frac{5}{8}+\frac{2}{5}*\frac{3}{8}}=\frac{25}{37}.\ \ D.$$

7. The only way that no boy can be seated adjacently to another boy is if boys and girls alternate. If we fix one seat at the table (as we could potentially rotate distinct seating arrangements), we see that there are 1*3*2*2*1*1 = 12 ways this could happen. Since there are 6!/6 = 120 distinct seating arrangements, the probability we arrive at one with the desired property is 12/120 = 1/10. B.

8. The matrix will be invertible iff AD-BC = 0, or if AD = BC. Taking the product of any 2 of {1,2,}, we could potentially achieve 1*1=1, 1*2=2, or 2*2=4. There are 2^4 = 16 ways in total to assign the values 1 or 2 to A, B, C, and D. We will count the number of ways that the matrix will not be invertible, and then take the complement in order to find the probability of invertibility. If AD = BC = 1, (A,B,C,D) = (1,1,1,1), and so there is 1 way this can happen. If AD = BC = 2, we could have (A,B,C,D) = (1,2,1,2), (2,1,2,1), (2,1,1,2), or (1,2,2,1) (4 ways). If AD = BC = 4, we must have (A,B,C,D) = (2,2,2,2) (1 way) In total, therefore, there are 1+4+1+ = 6 ways (out of 16) that the matrix would not be invertible. Therefore, the matrix will be invertible with probability
1 – 6/16 = 10/816 = 5/8. D.

9. The probability V has not occurred after 2 trials is (1 - $\dfrac{3-\sqrt{3}}{3}$)^2 = 1/3. C.

10. The events are independent, so the probability equals (.2)*(.4) = .08 = 8/100. C.

11. In order for 18 to be the maximum of T, we need 3 things to happen: 20 is not selected, 19 is not selected, 18 is selected. Half of the subsets do not contain 20, half of those do not contain 19, and half of those do contain 18. Therefore, the desired probability is 1/8. A.

12. There are 101 coefficients, with exponents on y ranging from 0 to 100. This is how we will index our terms. Term 0 has coefficient 1 (it is x^{100}), and term 100 has coefficient 1 as well. So, these terms are not divisible by 12. We know that the terms 2 through 99 must be divisible by 12, because they involve at least squaring (2x) and taking at least the first power of (3y), which automatically gives us a coefficient divisible by 12. So, the only term left to check is term 1. The coefficient on term 1 is 100*2*3, which is certainly divisible by 12. Therefore, there are only 2 coefficients out 101 which are not divisible by 12, so the desired probability is (101-2)/101 = 99/101. D.

13. We can think of the spinner as possibly having angles of incidence with the positive x-axis of [-90,90]. When this angle is in [-30,30], the displacement of the tips of the spinner from the x-axis will be exactly ½, since sin(30) = 1/2. Therefore, the desired probability is | [-30,30] | / |[-90,90] = 60/180 = 1/3. B.

14. The area of the inner circle is 36pi, and the area of the annulus is 144pi – 36pi = 108pi. Since the area of 4 points is 3 times greater than the area of the area of the area of 10 points, the probability that the dart hits the 4 point area is 3/(3+1) = 3/4, and the probability that the dart his the 10 point area is 1/(3+1) = 1/4. Therefore, the expected value is 10* (1/4) + 4* (3/4) = 5.5 A.

15. First we can find the point at which the total time would be exactly one minute. Then, we know that anywhere to the left of this point will have less 70 mph and more 50 mph, bringing the average time above one minute. Suppose the random point is x miles from the start of the ride. Then,

$$\text{Time} = x \text{ miles} * \frac{1\,\text{hour}}{70\,\text{miles}} + (1\text{-}x)\text{ miles} * \frac{1\,\text{hour}}{50\,\text{miles}} = \frac{x}{70} + \frac{1-x}{50} \text{ hours} = \frac{6x}{7} + \frac{6(1-x)}{5} \text{ minutes.}$$

So, setting time equal to 1, we see that $x = \frac{7}{12}$. If the point is anywhere from 0

to $\frac{7}{12}$ miles from the starting point, the trip will take longer than 1 minute.

Therefore, the answer is $\frac{7}{12}$. C.

16. Since P(A U B)^c = .1, P(A U B) = .9. Since P(A) = .55, P(only B) = .9 - .55 = .35. We do not know P(A $\cap$ B), but we know it could be as high as .35 (leaving p(only A) at .2), or as low as 0 (leaving p(only A) at .55). Inspection shows that when p(A U B) = 0, p(B) is forced to be .35. If p(A U B) is higher, p(B) is higher as well. Therefore, the smallest possible value of P(B) is 0.35. B.

17. There are 4 different combinations of the numbers 1 and 2 that sum to 6: 2+2+2, 2+2+1+1, 2+1+1+1+1, and 1+1+1+1+1+1. So, the probability that he at some point sums to 6 will be equal to the sum of the probability of rolling each of these sequences, times the number of ways it can be accomplished. Here, C(n,r) refers to the combination function, n!/[r! * (n-r)!]. The probability of rolling a sequence of n numbers will simply be (.5)^n, because P(1) = P(2) = .5. Each can be accomplished C(n,r) ways, where n is the number of digits and r is the number of 2's. Therefore, the total probability will be C(3,3)*(.5)^3 + C(4,2)*(.5)^4 + C(5,1)*(.5)^5 + C(6,0)*(.5)^6 = 43/64. D

18. Each of the 4 ants has a choice of going down 2 sides, so there are 2^4 = 16 ways the ants can travel. Only if they all go clockwise or counterclockwise will no paths cross. Therefore, there are 2 ways they won't intersect, and the probability no ants' paths intersect is 2/16 = 1/8. E.

19. There are C(10,2) = 45 different combinations of 2 distinct numbers that Marty can pick. The pairs {1,9}, {1,4}, {2,3}, {2,8}, {3,7}, {4,6}, and {5,10} have the desired property. Therefore, the answer is 7/45. C.

20. There are C(10,3) = 120 subsets with 3 elements. For the largest element to be less than or equal to 8, there must be no 9's or 10's. There are C(8,3) = 56 subsets that can be made using only the numbers {1,2,3,4,5,6,7,8}, and this is exactly what we would need for the highest element to be less than or equal to 8. Therefore, the probability is 56/120 = 7/15. B.

21. If there is an equal chance that they each win the game, then P(Jamal wins) = P(Josh wins) = .5. Let P(Josh makes a free throw) = p. Then, it must be that P(Josh Wins) = .5 = p*(1-.75) + p*.75*p*(1-.75) + p*.75*p*.75*p*(1-.75) + … In other words, in order for Josh to win, he must make a free throw and Jamal must eventually miss. The probability Josh wins is the sum of the probabilities that he wins after any number of go-arounds. We see that the equation on the right is an infinite geometric series, with first term p/4 and common ratio 3p/4. Therefore, ½ = (p/4) / (1 - 3p/4), and solving for p we see that p = 4/5. Therefore, the answer is .8. C.

22. Suppose Ross's position is given. Either he is seated in an end seat (with probability 2/10) or he is not (with probability 8/10). If Ross is seated in an end seat, there is a 1/9 chance that Jordan is seated next to him. If Ross is seated somewhere in the middle, then there is a 2/9 chance that Jordan is seated next to him (he can be seated on either side). Therefore, the probability that Ross and Jordan are seated next to one another is 2/10 * 1/9 + 8/10 * 2/9 = 18/90 = 1/5. B.

23. Suppose the center of the disc lands within a particular square on the sheet. The event of the disc landing on a line of the sheet occurs with probability zero, so our assumption is always true. If the center of the disc is at least 2 cm from each edge of the square, then the disc will not intersect any line on the sheet. This restricts the center of the disc to a smaller 2x2 square within the original square, which was by definition 6x6. Therefore, the probability the disc does not intersect any line on the sheet is 2*2/(6*6) = 1/9. D.

24. In order for the marlins to win the series, they must win the final game. They can win the series either by winning the first 2 games, or by winning only 1 of the first 2 games as well as the 3rd game. There are C(2,2) = 1 ways to accomplish the first of these, C(2,1) = 3 ways to accomplish the second. Therefore, the total probability is C(2,2)*P(win in 2) + C(2,1)*P(win in 3). When we plug numbers in, we obtain P(Marlins win Series) = 1 *(2/3)^2 + 2* (2/3)^2 * (1/3) = 20/27. C.

25. You have two chances to pick something with a probability of 1/8, therefore the answer is 2 * 1/8 = ¼. Alternatively, the probability can be determined by saying p(white) = p(white on 1st draw) + p(white on 2nd draw and black on first draw) = 1/8 + (1/7)(7/8) = 1/4. E.

26. In order for the price of the bar to be less than or equal to $1.00, the cost has to either decrease 4 times (wherein cost = $1.00*(.9)^4), decrease 3 times and raise

once (wherein cost = 1.00*(.9)^3*(1.1)), or decrease twice and raise twice (wherein cost = $1.00 * (.9)^2 * (1.1)^2 = 1.00*(.99)^2 < 1.00). A simple binomial expansion shows that the probability of the price ending up less than $1.00 is 11/16.   A.

27. In order to end up back at its starting point, the ant must move exactly once N, E, S, and W, or twice N and twice S, or twice E and twice W. This can happen in 4! +C(4,2) + C(4,2) = 36 different orders. There are a total of 4^4 = 256 combinations of movements the ant can make. Therefore, the probability is 36/256 = 9/64. B.

28. Given the quarter's position (head or tail), we'd require that at least 3 of the 4 pennies landed on that side. The probability of 3 of the 4 pennies landing on any one given side is C(4,3)/2^4 = ¼, and the probability of all 4 of the pennies landing on any given side is 1/16. Therefore, the desired probability is ¼ + 1/16 = 5/16.   C.

29. We need to solve P(lightning|rain)*p(rain) + p(lightning|no rain)*p(no rain) = p(lightning).   Now we can plug in terms noting that p(rain) = 1-p(no rain): .6p + .3(1-p) = .4.   Solving this equation for p, we see that p = 1/3.   C.

30. The number of zeroes at the end of a factorial will be determined by the number multiples of 5 in that product, because there will be ample 2s with which to multiply the 5s. Therefore, we can say that 30! is the lowest factorial that ends with 7 zeroes, because 30! includes (30)…(25)…(20)…(15)…(10)…(5), and 25 = 5^2. Therefore, 30! includes 7 factors of 5, which when multiplied by 7 factors of 2 will create 7 zeroes. Therefore, the integers X = {30, 31, 32, …, 100} satisfy the property that X! ends with at least 7 zeroes. Since there are 71 integers in this set, the probability desired is 71/100, or 0.71. A.