2007 Alpha School Bowl Solutions

Question #1

$$9x^2+16y^2+18x=64y+71 \rightarrow \quad 9x^2+18x+16y^2-64y=71 \rightarrow$$

$$9(x^2+2x+1)+16(y^2-4y+4)=71+9+64=144 \rightarrow \frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$$

$a=\sqrt{16}=4,\ b=\sqrt{9}=3$

A = area = $ab\pi=12\pi$

$c^2=a^2-b^2 \quad \rightarrow \quad c=\pm\sqrt{7}$ Foci: $\left(-1\pm\sqrt{7},\,2\right)$

B = sum of abscissas $=\left(-1+\sqrt{7}\right)+\left(-1-\sqrt{7}\right)=-2$

Center = (–1, 2) C = product of abscissa and ordinate: $-1\bullet 2=-2$

D = eccentricity = $\frac{c}{a}=\frac{\sqrt{7}}{4}$

$$A\bullet B\bullet C\bullet D=12\pi\bullet -2\bullet -2\bullet \frac{\sqrt{7}}{4}=12\pi\sqrt{7}$$

Question #2

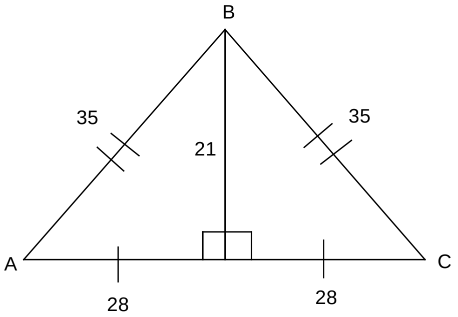
$R=10(20+0)-3(-5-0)+2(1-28)=161$

If only one column is multiplied by 2, then the original determinant is doubled: $S=2$

$8^n=2^{3n}$ will divide $44^{44}=2^{88}\bullet 11^{44}$ only when $3n\le 88$. The largest n is therefore 29, T.

Congruent triangles are formed by SSS.

$$\begin{aligned}x^2 &= 21^2+28^2\\ &= 441+784\\ &= \sqrt{1225}\\ x &= 35\\ P &= 35+35+56\\ &= 126\end{aligned}$$



$R+S+T+U=161+2+29+126=318$

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Question #3

$$\begin{cases} \dfrac{A+3}{4}+\dfrac{B-1}{3}=1 \\ 2A-B=12 \end{cases} \quad\to\quad \begin{cases} 3A+9+4B-4=12 \\ 2A-B=12 \end{cases} \quad\to\quad \begin{cases} 3A+4B=7 \\ 2A-B=12 \end{cases}$$

$$\left(A,B\right)=\left(5,-2\right)$$

$$\begin{cases} \dfrac{12}{C}-\dfrac{12}{D}=7 \\ \dfrac{3}{C}+\dfrac{4}{D}=0 \end{cases} \qquad \text{Let } x=\frac{1}{C},\ y=\frac{1}{D} \qquad \begin{cases} 12x-12y=7 \\ 3x+4y=0 \end{cases} \quad\to\quad \left(\frac{1}{3},-\frac{1}{4}\right)$$

$$\left(C,D\right)=\left(3,-4\right)$$

$\sin x\cos E+\cos x\sin E=\sin(x+E)\to\sin(22+E)=\dfrac{\sqrt{3}}{2}$. The angle with the required sine is 60°, so $E=38^\circ$.

$ABCDE=5\bullet-2\bullet3\bullet-4\bullet38=4560$

Question #4

$\left(x^3+1\right)^4\left(x^4+1\right)^5=\left(x^{12}+...\right)\left(x^{20}+...\right)=x^{32}+...$ The degree is 32, *A*.

$$\begin{array}{l} 7\,\underline{|298} \\ \quad\underline{|42}\ \ R4 \\ \qquad 6\ \ R0 \end{array} \qquad\qquad 298_{10}=604_7.\ B=604.$$

$320=20+(5-1)d\to300=4d\to d=75\Rightarrow a_{12}=320+2(75)=470, C$.

$${}_7P_3=7\bullet6\bullet5=210 \qquad \frac{1}{6!}=\frac{1}{720} \qquad {}_4C_2=\frac{4!}{2!2!}=6 \qquad {}_4P_2=4\bullet3=12$$

Diagonal matrix, so determinant = product of diagonal entries: $210\bullet\dfrac{1}{720}\bullet6\bullet12=21=D$

$B-A-C-D=604-32-470-21=81$.

Question #5

$$\begin{cases} 3(2)^3+m(2)^2-5(2)+n=0 \\ 3(-1)^3+m(-1)^2-5(-1)+n=0 \end{cases} \rightarrow \begin{cases} 24+4m-10+n=0 \\ -3+m+5+n=0 \end{cases} \rightarrow \begin{cases} 4m+n=-14 \\ m+n=-2 \end{cases} \rightarrow (-4,\ 2)$$

$$A=mn=-8$$

$$B=(-1+i)^{-4}=\left[\sqrt{2}\left(\cos 135^\circ+i\sin 135^\circ\right)\right]^{-4}\rightarrow\frac{1}{\left(\sqrt{2}\right)^4}\left[\cos\left(-540^\circ\right)+i\sin\left(-540^\circ\right)\right]$$

$$\rightarrow\frac{1}{4}(-1+0i)=-\frac{1}{4}$$

$ME=\frac{1}{2}CD$ because it is a median of triangle *ADC*. $NF=\frac{1}{2}CD$ because it is a median of triangle *BCD*.

$MN=\frac{1}{2}\left(CD+AB\right)$ because it is a median of the trapezoid.

$MN=\frac{1}{2}(8+20)=14;\ ME=\frac{1}{2}(8)=4=NF;\ 14-4-4=6=C$

If the vertical asymptote is at 4, then $c=4$. If horizontal asymptote is 2, then $a=2$ because leading coefficient on bottom is 1. If *y*-intercept is 1, then $1=\frac{2(0)+b}{(0)-4}\rightarrow -4=b$. *D* = sum of 2, –4, 4 = 2.

$$A\bullet B\bullet C\bullet D=-8\bullet-\frac{1}{4}\bullet 6\bullet 2=24$$

Question #6

$$\frac{\log_3 4+\log_3 8+\log_3 16+\log_3 32}{\log_3 2}=\frac{\log_3 4}{\log_3 2}+\frac{\log_3 8}{\log_3 2}+\frac{\log_3 16}{\log_3 2}+\frac{\log_3 32}{\log_3 2}$$

$$=\log_2 4+\log_2 8+\log_2 16+\log_2 32=2+3+4+5=14=A.$$

Semiperimeter of quadrilateral = $\frac{1+5+9+11}{2}=13$

By Brahmagupta's formula, area = $\sqrt{(13-1)(13-5)(13-9)(13-11)}=\sqrt{768}=B$. $B^2=768$.

$$\frac{\sqrt{0.005}}{\sqrt{2}}=\sqrt{\frac{\frac{5}{1000}}{2}}=\sqrt{\frac{5}{2000}}=\sqrt{\frac{1}{400}}=\frac{1}{20}=5\%.\ C=5.$$

After obtaining the common denominator and simplifying, $\frac{1}{3}\le\frac{x}{2007}\le\frac{4}{9}$ → $669\le D\le 892$.

The number of integers is 892 – 669 + 1 = 224, *D*.

The largest digit from *A*, *B*, *C*, and *D* is 8.

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Question #7

$$x^7+125x^4-x^3-125 \rightarrow x^4(x^3+125)-1(x^3+125) \rightarrow \left(x^4-1\right)\left(x^3+125\right)$$

$$\rightarrow (x^2+1)(x^2-1)(x+5)(x^2-5x+25)$$

$$\rightarrow (x^2+1)(x^2-5x+25)(x-1)(x+1)(x+5)$$

There are three linear factors in the factorization, so *A* = 3.

$$4^B-4^{B-1}=24 \rightarrow 4^B\left(1-4^{-1}\right)=24 \rightarrow 4^B\left(\frac{3}{4}\right)=24 \rightarrow 4^B=32 \rightarrow 2^{2B}=2^5$$

$$B=\frac{5}{2}$$

$\left|3x+7\right|\le 1\left(=5^0\right)$ $\rightarrow$ $3x+7\le 1$ and $3x+7\ge -1$ $\left[-\frac{8}{3},-2\right]$. –2 is the smallest integer in the solution set, *C*.

$$A\bullet B\bullet C=3\bullet\frac{5}{2}\bullet -2=-15.$$

Question #8

The prime factorization of $8400=2^4\bullet 3\bullet 5^2\bullet 7$. If a perfect square divides this product, it can have only 2 and 5 among its factors. It can only have zero, two, or four 2s and only zero or two 5s. Since the number is square it cannot have an odd number of any factor. Therefore, the number of ways is (3)(2) = 6.

$$\frac{x+7}{x^2-x-6}=\frac{B}{x-3}+\frac{C}{x+2}=\frac{B(x+2)}{(x-3)(x+2)}+\frac{C(x-3)}{(x-3)(x+2)}$$

$\rightarrow$ $x+7=B(x+2)+C(x-3)$

If x = 3: $3+7=B(3+2)+C(0) \rightarrow 10=5B \rightarrow B=2$

If x = –2, $-2+7=B(0)+C(-2-3) \rightarrow 5=-5C \rightarrow C=-1$.

$A=6, B=2, C=-1$ $\rightarrow$ $6\div 2\div -1=-3$.

Question #9

$$\begin{cases} y=3x^2-2x+5 \\ y=4x+2 \end{cases} \rightarrow 3x^2-2x+5=4x+2$$

$3x^2-6x+3=0$ $\rightarrow$ $x^2-2x+1=0$ $\rightarrow$ $(x-1)^2=0$

$x=1$

$y=4(1)+2=6$

Intersection: (1, 6) Distance = $\sqrt{(3-1)^2+(17-6)^2}=\sqrt{4+121}=\sqrt{125}=A$. $A^2=125$.

181 is the sum of two squares. 100 and 81 are fairly obvious. Checking with 10 and 9, we find that this pair works so *ab* = 90. Also, $(a+b)^2=a^2+2ab+b^2=19^2=361$. 2*ab* = 361 – 181 = 180, so *ab* = 90, *B*.

$64=2^6$ so it is a sixth, a cube, a square, and a first power.

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If $\begin{cases} z=6: (x,y)=(2,1) \\ z=3: (x,y)=(4,1),\ (2,2) \\ z=2: (x,y)=(8,1), (2,3) \\ z=1: (x,y)=(64,1), (8,2), (4,3), (2,6) \end{cases}$ There are C = 9 possible triples.

The units digits to the above answers are 5, 0, and 9. Their sum is 14.

Question #10

When evaluated at $x=1$, $\lim\limits_{x\to1}\frac{\sqrt[3]{x}-1}{x-1}=\frac{0}{0}$. $\frac{\sqrt[3]{x}-1}{x-1}=\frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$.

Therefore, $\lim\limits_{x\to1}\frac{\sqrt[3]{x}-1}{x-1}=\lim\limits_{x\to1}\frac{1}{\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$. Evaluating again at $x=1$, $\lim=\frac{1}{3}=\frac{A}{B}$.

$\sqrt{PE\bullet ET}=16\to(PE)(ET)=256$.

$Area=84=\frac{1}{2}(PE)(ET)\sin P$

$\sin P=\frac{21}{32}=C$

$1+3+2+1+3+2=12$

Question #11

$\log_{\cos x}\sin x=\frac{1}{2}\to\left(\cos x\right)^{\frac{1}{2}}=\sin x$

$\cos x=\sin^2 x\to\cos x=1-\cos^2 x$

$\cos^2 x+\cos x-1=0$

$\left[\cos x\right]=\left[\frac{-1\pm\sqrt{1-4(1)(-1)}}{2(1)}\right]=\left[\frac{-1+\sqrt{5}}{2}\right]=A\approx\left[\frac{-1+2.2}{2}\right]=0$

The other value of cos*x* is extraneous since *x* must be in Quadrant I.

Using sum and product of roots, product $=\sin u\cos u=\frac{k}{9}$ and sum $=\sin u+\cos u=\frac{2}{9}$.

$\sin u+\cos u=\frac{2}{9}\to(\sin u+\cos u)^2=\sin^2 u+\cos^2 u+2\sin u\cos u=\frac{4}{81}$.

$=1+2\left(\frac{k}{9}\right)=\frac{4}{81};k=-\frac{77}{18}$

$A+B=0-\frac{77}{18}=-\frac{77}{18}$.

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Question #12

$e \approx 2.72, \pi \approx 3.14 \quad \rightarrow \quad e + \pi \approx 5.86 \Rightarrow 6.$

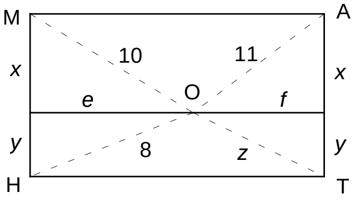
There are fewer 5's in 40! than 2's, so divide:

$$\begin{array}{r} 8 \\ 5\overline{)40} \end{array} \qquad \begin{array}{r} 1 \\ 5\overline{)8} \end{array} \qquad 8+1=9=B$$

$$x^3 + x - 8 = \frac{8}{x^2} \quad \rightarrow \quad x^5 + x^3 - 8x^2 - 8 = 0 \quad \rightarrow \quad x^3\left(x^2+1\right) - 8\left(x^2+1\right) = 0$$

$$\rightarrow \quad \left(x^2+1\right)\left(x^3-8\right) = 0 \quad \rightarrow \quad \left(x^2+1\right)\left(x-2\right)\left(x^2+2x+4\right) = 0$$

x^2+1 and x^2+2x+4 only yield only imaginary solutions. There is only one real solution, so $C = 2$.



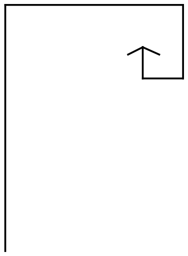
$$\left.\begin{array}{l} x^2 + e^2 = 10^2 \\ y^2 + e^2 = 8^2 \end{array}\right\} \rightarrow 100 - x^2 = 64 - y^2$$

$$\left.\begin{array}{l} x^2 + f^2 = 11^2 \\ y^2 + f^2 = z^2 \end{array}\right\} \rightarrow 121 - x^2 = z^2 - y^2$$

$$\rightarrow 21 = z^2 - 64 \Rightarrow z = \sqrt{85}. \; D = 9$$

The answer choices are 6, 9, 2, and 9. The mode is 9.

Question #13



The common ratio for each move is $-\frac{1}{9}$, as the first south move is two moves after the first north move (likewise for east-west) and each move is in the opposite direction as the move two steps before. The first east-west move is 60 m. Therefore, the total distance traveled each direction is an infinite geometric series.

North-South: $\dfrac{180}{1-\left(-\frac{1}{9}\right)} = 162$ East-West: $\dfrac{60}{1-\left(-\frac{1}{9}\right)} = 54$

The final stop for the armadillo is the point (54, 162) = (*A*, *B*).

$$x^{\log x} = 100x \quad \rightarrow \quad \log x^{\log x} = \log 100x$$

$$\rightarrow \left(\log x\right)^2 = \log 100 + \log x \quad \rightarrow \quad \left(\log x\right)^2 - \log x - 2 = 0$$

$$\rightarrow (\log x - 2)(\log x + 1) = 0 \quad \rightarrow \quad \log x = 2, \log x = -1$$

$$x = 100, x = \frac{1}{10} \quad \Rightarrow \quad C = 100$$

$$\frac{B-A}{C} = \frac{162-54}{100} = \frac{108}{100} \rightarrow 108\%$$

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Question #14

$x = \sqrt{7+2\sqrt{6}} - \sqrt{7-2\sqrt{6}} = a - b$.

$$\begin{aligned}(a-b)^2 &= a^2 - 2ab + b^2 \\ &= \left(\sqrt{7+2\sqrt{6}}\right)^2 - 2\left(\sqrt{7+2\sqrt{6}}\right)\left(\sqrt{7-2\sqrt{6}}\right) + \left(\sqrt{7-2\sqrt{6}}\right)^2 \\ &= 7 + 2\sqrt{6} - 2\sqrt{49-24} + 7 - 2\sqrt{6} \\ &= 14 - 2\sqrt{25} \\ &= 14 - 10 \\ &= 4 = A\end{aligned}$$

Since $990 = 2 \bullet 3^2 \bullet 5 \bullet 11$, *B*! must contain a factor of 11. Since 11 is prime, 11! Is the smallest factorial that contains a factor of 11, so *B* = 11.

$A + B = 15$.

Question #15

$1{,}000{,}000{,}000 = 10^9 = 2^9 \bullet 5^9 = 1{,}953{,}125 \bullet 512 \quad \rightarrow \quad A = 512$

Let $B = x^2$, $n = 43$.

$x^2 + n = y^2 \quad \rightarrow \quad n = y^2 - x^2 = (y+x)(y-x)$

$\quad \rightarrow \quad 43 = (y+x)(y-x)$

This occurs when $y - x = 1, y + x = 43 \quad \rightarrow \quad x = 21, y = 22$.

$21^2 + 43 = 22^2$ so the desired square number *B* is $B = 21^2 = 441$.

TAMPA has 5 letters but 2 A's, so the desired set-up is $\frac{5!}{2!} = 5 \bullet 4 \bullet 3 = 60$.

$A + B + C - 1 = 512 + 441 + 60 - 1 = 1012$.