Alpha State Bowl 2007 Mu Alpha Theta National Convention
Solutions

1\. A $252 = 2^2 \bullet 3^2 \bullet 7^1$ $\quad 308 = 2^2 \bullet 7^1 \bullet 11^1$ $\quad 504 = 2(252) = 2^3 \bullet 3^2 \bullet 7^1$

$GCF = 2^2 \bullet 7^1 = 28$. The largest prime less than 28 is 23.

B When multiplying the first *a* with the second expression, 9 terms are produced. When multiplying the first *b* with the second expression, 8 terms are produced. Each product will yield one less term than the previous. Thefore, multiplication by the first *j* will yield only 1 term. The sum of the integers 1 through 9 is 45.

C $x = \log_{\left(\frac{3^2 \cdot 22222^2 + 99999^2}{11111^2 \cdot 3^2}\right)} \left(2^2 + 3^2\right)^5 = \log_{\left(\frac{3^2 \bullet 1111^2 \bullet 2^2 + 1111^2 \bullet 3^2 \bullet 3^2}{11111^2 \cdot 3^2}\right)} \left(2^2 + 3^2\right)^5$

$= \log_{\left(2^2+3^2\right)} \left(2^2 + 3^2\right)^5 = 5$. $x^2 + 1 = 26$.

D The 1004th integer is the middle term, *x*, of the 2007 integers beginning with $x - 1003, x - 1002, \ldots, x, \ldots, x + 1002, x + 1003$. Adding the terms we get $2007x = 22{,}077$, so $x = 11$.

2\. A $\dfrac{3x+25}{2x-5}$ is not evenly divisible. Double the numerator and denominator to obtain $\dfrac{6x+50}{2x-5} \rightarrow 3 + \dfrac{65}{2x-5}$. For this new fraction to be evenly divisible, the denominator must be a divisor of 65, namely $\pm 1, \pm 5, \pm 13, \pm 65$. The values of *x* obtained from these eight numbers are 3, 2, 5, 0, 9, –4, 35, and –30. Their sum is 20.

B Using column 3: $-3(14x - 5x) - 0 + 1(5 - (-14)) = -27x + 19 = -35$.
$-27x = -54 \rightarrow x = 2$.

C $x^2 + y^2 + 4x - 46 = 0$ becomes $(x+2)^2 + y^2 = 50$. Center: (–2, 0), point on circle: (3, 5). Slope between points is $m = \dfrac{5-0}{3-(-2)} = 1$. Since the tangent line is perpendicular to the radius, $m_{\perp} = -1$. Now, $y - 5 = -1(x - 3) \rightarrow y = -x + 8$

D In each hour from 1:00 through 9:59, there are six such times. For example, during the 2:00 hour, there are 2:02, 2:12, 2:22, 2:32, 2:42, and 2:52. From 10:00 through 12:59, there is only one time each hour, such as 10:01. The total number of times is 9(6) + 3 = 57.

3\. A Using $\tan 2x = \dfrac{\sin 2x}{\cos 2x} = \dfrac{2\sin x \cos x}{\cos 2x}$, we get $\dfrac{2\sin x \cos x}{\cos 2x} + 2\sin x = 2\sin x\left(\dfrac{\cos x}{\cos 2x} + 1\right)$

$= 2\sin x\left(\dfrac{\cos x + \cos 2x}{\cos 2x}\right) = 0$. If $\sin x = 0$, then $0, \pi$. For $\cos x + \cos 2x = 0$, we get

$\cos x + 2\cos^2 x - 1 = (2\cos x - 1)(\cos x + 1) = 0$, so $x = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$. The sum of all solutions is 3π.

B There are 2 choices for the driver, 5 for the front seat passenger, and 4 for the back seat passenger for a total of $2 \bullet 5 \bullet 4 = 40$ ways.

C The unknown *x* appears in the same polynomials in both the expressions (free of radicals and under the radical). Add 4 to each side:

$3x^2 - 5x + \sqrt{3x^2 - 5x + 4} = 16 \rightarrow 3x^2 - 5x + 4 + \sqrt{3x^2 - 5x + 4} = 20$.

Let $y = \sqrt{3x^2 - 5x + 4}$. We now have $y^2 + y - 20 = (y+5)(y-4) \rightarrow y = -5, 4$.

y cannot equal –5, therefore $\sqrt{3x^2 - 5x + 4} = 4 \rightarrow 3x^2 - 5x + 4 = 16 \rightarrow$ $(3x+4)(x-3) = 0$. The only positive solution for *x* is 3.

D Rearranging terms, $\frac{\log_3 5 \bullet \log_7 3 \bullet \log_2 7}{\log_{11} 5 \bullet \log_{17} 11 \bullet \log_8 17} \rightarrow \frac{\log_3 3 \bullet \log_7 7 \bullet \log_2 5}{\log_{11} 11 \bullet \log_{17} 17 \bullet \log_8 5} \rightarrow \frac{\log_2 5}{\log_8 5}$

$\frac{\log_5 8}{\log_5 2} \rightarrow \log_2 8 = 3$.

4. A $10 = -(x-6)^2 + 36 \rightarrow 26 = (x-6)^2 \rightarrow x = 6 \pm \sqrt{26}$. The only solution where the ball would be above the net is $x = 6 + \sqrt{26}$.

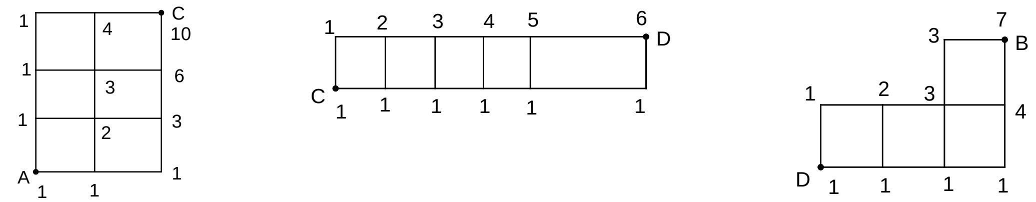
B $$\frac{\cos A \cot A - \sin A \tan A}{\csc A - \sec A} \rightarrow \frac{\cos A \frac{\cos A}{\sin A} - \sin A \frac{\sin A}{\cos A}}{\frac{1}{\sin A} - \frac{1}{\cos A}} \rightarrow \frac{\cos^3 A - \sin^3 A}{\cos A \sin A}$$

$$\rightarrow \frac{(\cos A - \sin A)(\cos^2 A + \cos A \sin A + \sin^2 A)}{\cos A - \sin A} \rightarrow \cos^2 A + \cos A \sin A + \sin^2 A$$

$$\Rightarrow 1 + \sin A \cos A \text{ or } 1 + \frac{1}{2}\sin\frac{A}{2}$$

C $\frac{6!}{3!3!}\left(\frac{\sqrt{x}}{y^2}\right)^3 \left(\frac{-y}{\sqrt{x}}\right)^3 = 20 \bullet \frac{x\sqrt{x}}{y^6} \bullet \frac{-y^3}{x\sqrt{x}} = -\frac{20}{y^3}$.

D Eventually, one of the terms will be ($x - x$), which is 0, so the product is 0.

5. A 

There are 10 ways to get from *A* to *C*. There are 6 ways to get from *C* to *D*. There are 7 ways to get from *D* to *B*. Multiplying these numbers gives 420.

B Let $A=\text{Cot}^{-1}\frac{3}{7}, B=\text{Cot}^{-1}\frac{1}{4}$ so that $\tan A=\frac{7}{3}, \tan B=\frac{1}{4}$. Using the tangent difference formula, $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{\frac{7}{3}-\frac{1}{4}}{1+\left(\frac{7}{3}\right)\left(\frac{1}{4}\right)}=\frac{25}{19}$.

C $\left(3^{9x+11}\right)\left(7^{5x+9}\right)\left(4^{10x+18}\right)=\left(336^{a}\right)\left(3^{b}\right)$

$$\left(3^{9x+11}\right)\left(7^{5x+9}\right)\left(16^{5x+9}\right)=\left(21^{a}\right)\left(16^{a}\right)\left(3^{b}\right)=\left(7^{a}\right)\left(3^{a}\right)\left(16^{a}\right)\left(3^{b}\right)$$
$$=\left(7^{a}\right)\left(3^{a+b}\right)\left(16^{a}\right)$$

$3^{a+b}=3^{9x+11} \quad \Rightarrow \quad a+b=9x+11$

D Using $D=\frac{n(n-3)}{2}$, $299=\frac{n(n-3)}{2} \rightarrow 598=n^2-3n \rightarrow (n-26)(n+23)=0$.
The number of sides is 26.

6. A Find the probability of a bronze trophy being drawn from box *X* and a gold trophy being drawn from box *Y* or a gold trophy being drawn from box *X* and a gold trophy being drawn from box *Y*: $\left(\frac{4}{9}\right)\left(\frac{8}{20}\right)+\left(\frac{5}{9}\right)\left(\frac{9}{20}\right)=\frac{8}{45}+\frac{1}{4}=\frac{77}{180}$.

B Even factors = Total factors – odd factors.
$3450=2^1\bullet 3^1\bullet 5^2\bullet 23^1$. $(1+1)(1+1)(1+2)(1+1)=24$ total positive factors.
Number of positive odd factors: $3^1\bullet 5^2\bullet 23^1 \rightarrow (1+1)(1+2)(1+1)=12$.
Total – odd = 24 – 12 = 12. Since positive factors were not specified, there are 24 factors, including the negative factors.

C $\sin\frac{\pi}{2}=1$, so the problem is $y=\sqrt{1+\sqrt{1+\sqrt{1+...}}}$. Squaring each side gives $y^2=1+y \rightarrow y^2-y-1=0 \rightarrow y=\frac{1\pm\sqrt{5}}{2}$. Answer must be positive, so $y=\frac{1+\sqrt{5}}{2}$.

D With 4 in the hundreds places, the last two digits taken as one number must be divided by 8. Since we are not allowed to use 4, 0, or double digits, we have only five choices for the last two positions: 16, 32, 56, 72, and 96. That leaves six numbers to choose from for the previous six positions. The total therefore is $5 \bullet 6! = 3600$.

7. A If the sum is prime, then the sum must be 2, 3, 5, 7, 11, or 13. Those probabilities are $\frac{1}{64}+\frac{2}{64}+\frac{4}{64}+\frac{6}{64}+\frac{6}{64}+\frac{4}{64}=\frac{23}{64}$.

B $\frac{1}{x}+\frac{1}{y}=\frac{1}{2} \rightarrow 2y+2x=xy \rightarrow y=\frac{2x}{x-2}=2+\frac{4}{x-2}$. For the fraction to become an integer, x must be 6, 4, 3, 1, 0, or -2. 0 creates an undefined expression, so only five possibilities exist.

C
$$(x+3)\left(x-\frac{7}{6}\right)+\left(x-\frac{7}{6}\right)\left(x-\frac{14}{11}\right)=0 \rightarrow \left(x-\frac{7}{6}\right)\left(x+3+x-\frac{14}{11}\right)=0$$
$$\rightarrow \left(x-\frac{7}{6}\right)\left(2x+\frac{19}{11}\right)=0 \rightarrow x=\frac{7}{6},-\frac{19}{22} \Rightarrow Q=-\frac{19}{22}$$

D Taking the log of each side gives $(\log x)(\log x)=\log 1000x^2=3+\log x^2$.
This becomes $(\log x)^2-2\log x-3=0 \rightarrow (\log x-3)(\log x+1)=0$.
The two solutions for x are 1000 and 0.1, so their product is 100.

8. A $87!+88!=87!+88(87!)=87!(1+88)=87!(89)$. The largest prime is therefore 89.

B $78.4=\frac{784}{10}=\frac{392}{5}=\frac{8\bullet 49}{5}=\frac{2^3\bullet 7^2}{5}$, so $\log 78.4=\log\frac{2^3\bullet 7^2}{5}$. This becomes $\log 2^3+\log 7^2-\log 5=3\log 2+2\log 7-\log 5=3M+2P-N$.

C The ellipse $\frac{x^2\pi^2}{9}+\frac{y^2}{144}=1$ has area $\left(\frac{3}{\pi}\right)(12)\pi=36$. The number of handshakes therefore is ${}_{36}C_2=\frac{36!}{34!2!}=\frac{36\bullet 35}{2}=18\bullet 35=630$.

D Using Heron's formula, $s=\frac{13+14+15}{2}=21$, $A=\sqrt{21(21-13)(21-14)(21-15)}$ $=\sqrt{21\bullet 8\bullet 7\bullet 6}=84$.

9. A $3\tan\frac{x}{2}+3=0 \rightarrow \tan\frac{x}{2}=-1$. If $\frac{x}{2}=135$, $x=270$. If $\frac{x}{2}=315$, $x=630$. 270 is the only angle on our given interval.

B One solution to $x!=20y!$ is, by sight, (20, 19). Needing consecutive positive integers with a product of 20, we notice that $20=5\bullet 4$. Therefore, our second solution is (5, 3).

C $(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz = x^2+y^2+z^2+2(xy+xz+yz)$.
$=3+2(3)=9$. Therefore, $x+y+z=\pm 3$.

D Since 6 is a digit, we know the base must be greater than 6. Noting that in the problem $4 \bullet 5$ ended in 6 and that 20 ends with a six in base 7, 14, etc., and also that the answer is greater than it would be in base 10, the base must be less than 10. Therefore, the base is 7.

10. A Only two numbers are their own square roots: 0 and 1. If $5-\frac{1}{x}=0$, $x=\frac{1}{5}$.
If $5-\frac{1}{x}=1$, $x=\frac{1}{4}$. The product of the two numbers is $\frac{1}{20}$. Also, by squaring each sides of the given equation, the result is $20x^2-9x+1=0$, which is factorable and will give the same solutions.

B $5y-3xy = y(5-3x) = -6x^2+13x-11$. $y=\frac{-6x^2+13x-11}{5-3x} = 2x-1+\frac{6}{3x-5}$.
For y to be an integer, x must be 1 or 2. The possible ordered pairs are (1, –2) and (2, 9). The maximum product is obtained from (2, 9), and that product is 18.

C Add the exponents on the left and equate the sum to the exponent on the right side to obtain $\sin^2 x+\cos^2 x+\tan^2 x=2 \rightarrow \tan^2 x-1=0$. Upon factoring, the only factor that results in a positive-angle solutions is $\tan x-1=0$, which yields $x=\frac{\pi}{4}$.

D A maximum of two pieces can be made with one cut, a maximum of four pieces with two cuts, and a third cut can divide a maximum of three of the existing cuts into two pieces, for a total of seven pieces. A fourth cut can divide a maximum of four of the existing cuts into two pieces for a total of eleven pieces. The formula that results is the formula for a “cake number” or a “lazy caterer’s number”: $\frac{n^2+n+2}{2}$, where n is the number of cuts.