Calculus Hustle 2007 Solutions
Mu Alpha Theta National Convention

1. To find the values at which the particle isn't moving, one must find a value of t for which all the derivatives of the functions are zero.
$x'(t) = 12t^2 + 12t - 24 = 12(t^2 + t - 2) = 12(t+2)(t-1)$
$y'(t) = 6t^2 + 6t - 12 = 6(t^2 + t - 2) = 6(t+2)(t-1)$
$z'(t) = 3t^2 + 12t + 12 = 3(t^2 + 4t + 4) = 3(t+2)^2$
As can be seen by the factoring of the derivatives, $\boxed{t = -2}$ is the only simultaneous solution to the derivatives of the movement functions and therefore is the only answer.

2. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a well-known identity (based on a Maclaurin expansion) to most pre-calculus students. The given summation is very similar to the identity with $x = i\pi$, the difference between the identity and summation is the summation starts at $n = 1$. To put it in the proper form:
$$\sum_{n=1}^{\infty} \frac{(i\pi)^n}{n!} = \frac{(i\pi)^0}{0!} - \frac{(i\pi)^0}{0!} + \sum_{n=1}^{\infty} \frac{(i\pi)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i\pi)^n}{n!} - \frac{(i\pi)^0}{0!} = e^{i\pi} - 1 = -1 - 1 = \boxed{-2}$$

3. Differentiating the given equation and then substituting the given point:
(NOTE $y' = dy/dx$)
$3x^2 + 3y^2y' + 6xy + 3x^2y' - 6x + 3y^2 + 6xyy' = 0$
$12 + 3y' - 12 + 12y' - 12 + 3 - 12y' = 0$
$3y' = 9$
$\boxed{y' = 3}$
<u>or</u> simplifying first, the original equation can be rewritten as $(x+y)^3 = 3x^2 - 11$, differentiating this equation gives $3(x+y)^2(1+y') = 6x$, substituting and solving for y' gives us $\boxed{y' = 3}$.

4. $f(x) = \frac{x^4 - 2x^2 + 1}{x^3 - x} = \frac{(x^2-1)^2}{x(x^2-1)} = \frac{(x^2-1)}{x} \cdot \frac{(x-1)(x+1)}{(x-1)(x+1)}$. As can be seen $x = 0$ is an infinite discontinuity and $x = \pm 1$ are removable discontinuities. So $A = 1$, $B = 2$ and $A - B = \boxed{-1}$

5. First we must maximize the base (bottom) of the box. In general $P = 2w + 2l$, $A = wl$ for a rectangle, letting P be constant and trying to maximize A, you will find that $w = l$, hence the base of the box is a square. Let s be the length of one side of the square and h be the height of the box. We have that $24 = s^2 + 4hs$ and we are trying to maximize the equation $V = hs^2$. Solving for h in the first and substituting into our equation for V, we get $V = \frac{(24 - s^2)}{4s} s^2 = 6s - \frac{s^3}{4}$.

Differentiating with respect to s, $\frac{dV}{ds} = 6 - \frac{3s^2}{4}$. Now we want the value of s for which $\frac{dV}{ds}$ is 0, which is $s = 2\sqrt{2}$, plugging into our equation for V and simplifying, we get $V = \boxed{8\sqrt{2}}$

6. Let $y = \sqrt{x}$, then $dy = \frac{dx}{2\sqrt{x}}$. Substituting $dx = 14 - 16 = -2$ and $x = 16$ we get that $dy = -\frac{1}{4}$, and therefore we approximate $\sqrt{14} = y + dy = 4 - \frac{1}{4} = \boxed{\frac{15}{4}}$

7. $A = \pi r^2$, $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. $r = r_o + t\frac{dr}{dt} = 1 + 3 \cdot 3 = 10$. So $\frac{dA}{dt} = 2\pi(10)(3) = \boxed{60\pi}$

8. $A = \begin{pmatrix} \overline{\overline{\frac{1}{2}e^{2\cdot \ln x}}} & 2x-3 & x-1 \\ \overline{0} & \overline{3} & \overline{\overline{x^2}} \\ \overline{-2} & \overline{\overline{\ln(4x)}} & (2x-3)^3 \end{pmatrix}$, those entries in A marked with a single overbar will all be a zero in B. Therefore those entries in A marked with a double overbar are all that are needed to establish the value of the determinant. (To see why, we use expansion by minors along the first column and note the determinant of the 2x2 that matters has a zero along one diagonal). We can simplify the first entry as $\frac{1}{2}e^{2\cdot \ln x} = \frac{1}{2}x^2$. When we take derivatives of the double overbar marked entries and multiply them together (with a negative sign, to arrive at the determinant of B) we get $-(x)(2x)\left(\frac{1}{x}\right) = \boxed{-2x}$.

9. $f(x) = \sin(ix)$, $f'(x) = i\cos(ix)$, $f''(x) = -i^2\sin(ix) = \sin(ix)$, so $f(x) = f''(x)$ and therefore $f^{(19)}(x) = f'(x) = \boxed{i\cos(ix)}$

10. $v(t) = \int a(t)dt = \int \cos\left(\frac{t}{2}\right)dt = 2\sin\left(\frac{t}{2}\right) + C$.

$0 = v(0) = 2\sin\left(\frac{0}{2}\right) + C = C$, $\quad v(t) = 2\sin\left(\frac{t}{2}\right)$. Speed is the absolute value of velocity. So $|v(3\pi)| = \left|2\sin\left(\frac{3\pi}{2}\right)\right| = |-2| = \boxed{2}$

11. Let $h = 3j$, then as $h \to \infty$, $\quad j \to \infty$. Substituting into our equation we get

$$\lim_{j\to\infty}\left(1 + \frac{3}{3j}\right)^{3j} = \lim_{j\to\infty}\left(\left(1 + \frac{1}{j}\right)^{j}\right)^3 = \boxed{e^3}$$

12. $\frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}$, so $\int_3^6 \frac{2}{x^2-1}dx=\int_3^6\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\ln|x-1|-\ln|x+1|\Big]_3^6=$
$(\ln 5-\ln 7)-(\ln 2-\ln 4)=\ln\frac{5\cdot 4}{7\cdot 2}=\boxed{\ln\frac{10}{7}}$

13. $g(x)=\int f'(f(x))\cdot f'(x)dx=f(f(x))+C=(x^2+1)^2+1+C=$
$x^4+2x^2+1+1+C=x^4+2x^2+2+C=g(x)$
$2=g(0)=2+C,\ C=0,\ \boxed{g(x)=x^4+2x^2+2}$

14. $f'(x)=2e^{2x}-10e^x+12=2(e^{2x}-5e^x+6)=2(e^x-3)(e^x-2)$ Horizontal tangent lines occur when $f'(x)=0$, so when $x=\ln 2$ or $\boxed{x=\ln 3}$.

15. $f'(x)=x^2+2x-3=(x+3)(x-1)$ and $f''(x)=2x+2=2(x+1)$. Therefore $f(x)$ is decreasing when $-3<x<1$ and concave up whenever $x>-1$. Combining these two inequalities we get $-1<x<1$, which is rewritten as $\boxed{x\in(-1,1)}$

16. $A=\int_{-\pi}^{\pi}|\sin(x)|=2\int_0^{\pi}\sin(x)$, we originally need the absolute value bars since we're dealing with area, we remove them afterward since sin is positive on the new interval. $A=2\int_0^{\pi}\sin(x)=2[-\cos(x)]_{x=0}^{x=\pi}=2[-\cos(\pi)+\cos(0)]=2(1+1)=\boxed{4}$

17. Let $u=\tan x$, then $du=\sec^2 x\,dx$, so
$\int\frac{\sec^2 x}{1+\tan^2 x}dx=\int\frac{1}{1+u^2}du=\arctan u+C=\arctan(\tan x)+C=\boxed{x+C}$
or $\int\frac{\sec^2 x}{1+\tan^2 x}dx=\int\frac{1+\tan^2 x}{1+\tan^2 x}dx=\int 1dx=\boxed{x+C}$

18. Method 1 (AB Calc): This parametric "curve" is really a straight line from $(x(0),y(0))=(0,0)$ to $(x(2),y(2))=(12,16)$, since both $x(t)$ and $y(t)$ are dependent on the same variable (t^2). Therefore the "arc" length is
$\sqrt{(12-0)^2+(16-0)^2}=\sqrt{12^2+16^2}=4\sqrt{3^2+4^2}=4\cdot 5=\boxed{20}$
Method 2 (BC Calc): Definition of arc length for a parametrically defined curve is
$s=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$. Plugging in all of our values we get
$s=\int_0^2\sqrt{(6t)^2+(8t)^2}dt=\int_0^2 t\left(\sqrt{6^2+8^2}\right)dt=\int_0^2 10t\,dt=5t^2\Big]_{t=0}^2=\boxed{20}$

19. Let $u=-x^2$, then $du=-2xdx\quad\Rightarrow\quad -\frac{du}{2}=xdx$. Substituting we get
$\int_0^{\infty}xe^{-x^2}dx=-\frac{1}{2}\int_0^{-\infty}e^u du=-\frac{1}{2}e^u\Big]_{u=0}^{-\infty}=-\frac{1}{2e^{x^2}}\Big]_{x=0}^{\infty}=-0+\frac{1}{2}=\boxed{\frac{1}{2}}$

20. $f'(x)=-3x^2+3=-3(x^2-1)=-3(x-1)(x+1)$, so the points we need to look at are $x=\pm 1$ (the critical points) and $x=\pm 3$ (the endpoints).

x	-3	-1	1	3
$f(x)$	19	-1	3	-17

Glancing at the table we see the maximum value of the function is $\boxed{19}$.

21. Using the chain rule 3 times, $\dfrac{d\left[\sin^2\left(x^2\right)\right]}{dx} = 2\sin\left(x^2\right)\cos\left(x^2\right)\cdot(2x) = \boxed{2x\sin\left(2x^2\right)}$

22. $f\left(g(x)\right) = x$, differentiating both sides with respect to x yields,
$f'\left(g(x)\right)\cdot g'(x) = 1$, which is equivalent to $f'\left(g(x)\right) = \dfrac{1}{g'(x)}$, since we are looking for $f'(2)$, we need to know for what value of x is $g(x) = 2$, since $g(x) = f^{-1}(x)$, $x = 3$. Plugging in $x = 3$ to our formula above and using the given fact that $g'(3) = 11$, we get the answer of $\boxed{\dfrac{1}{11}}$.

23. $\lim\limits_{h\to 0}\dfrac{(4+h)^2-(4-h)^2}{2h} = \lim\limits_{h\to 0}\dfrac{\left(16+8h+h^2\right)-\left(16-8h+h^2\right)}{2h} = \lim\limits_{h\to 0}\dfrac{16h}{2h} = 8$ <u>or</u>
recognize the limit as the definition of derivative where the function is x^2 and it is being evaluated at $x = 4$. (Note: This is an alternate definition of the limit normally used to represent a derivative, changed by approaching infitesimally close from both directions, instead of one direction and the point itself.)

24. <u>Method 1</u> (AB Calc): $\lim\limits_{x\to 0}\dfrac{\sin(x)-\cos(x)+1}{x^3-3x^2+3x} = \lim\limits_{x\to 0}\dfrac{\sin(x)-\cos(x)+1}{x\left(x^2-3x+3\right)} =$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}-\frac{\cos x-1}{x}\right)\left(\frac{1}{x^2-3x+3}\right) = (1-0)\left(\frac{1}{0-0+3}\right) = \boxed{\frac{1}{3}}$$

<u>Method 2</u> (L'Hopital's Rule):

$$\lim_{x\to 0}\frac{\sin(x)-\cos(x)+1}{x^3-3x^2+3x} = \lim_{x\to 0}\frac{\cos(x)+\sin(x)}{3x^2-6x+3} = \frac{1+0}{0-0+3} = \boxed{\frac{1}{3}}$$

25. $f'(x) = -3(-x-1)^2(x+1)^2(2x-3)+2(-x-1)^3(x+1)(2x-3)+2(-x-1)^3(x+1)^2$
$f'(1) = -3(-2)^2(2)^2(-1)+2(-2)^3(2)(-1)+2(-2)^3(2)^2 = 16(3+2-4) = \boxed{16}$