Pre-Cal Hustle 2007 Solutions

1. There is a theorem that states that all solutions to the equation $z^6 = 1$ lie on the unit circle (of the complex plane). Therefore we may write z as $e^{i\theta}$ (since $r = 1$ on the unit circle). Now rewriting the equation $\left(e^{i\theta}\right)^6 = e^{6i\theta} = 1$, which is true whenever $6\theta = 2\pi k$ for some integer k, that is $\theta = \frac{\pi k}{3}$ which will give unique (and all) solutions for $k = 0,1,2,3,4,5$. The one in the third quadrant which we are looking for occurs when $k = 4$. So the solution of the equation is $e^{\frac{4\pi}{3}i} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = \boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}$

2. $\frac{3+2i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{3+9i+2i-6}{1-9i^2} = \frac{-3+11i}{10} = \boxed{\frac{-3}{10} + \frac{11}{10}i}$

3. $\left(e^3\right)^{\ln 2} \cdot \ln\left(\frac{\sqrt{e}}{e^2}\right) = \left(e^{3\ln 2}\right) \cdot \left[\ln\left(\sqrt{e}\right) - \ln\left(e^2\right)\right] = e^{\ln 8}\left(\frac{1}{2} - 2\right) = 8\left(\frac{1}{2} - 2\right) = 4 - 16 = \boxed{-12}$

4. $\sin(2x) - \sin x = 2\sin x\cos x - \sin x = \sin x(2\cos x - 1) = 0$, so either $2\cos x - 1 = 0 \quad \Rightarrow \quad \cos x = \frac{1}{2}$ in which case x can be $\frac{\pi}{3}$ or $\frac{5\pi}{3}$ on our interval, or $\sin x = 0$ in which case x can be π or 2π, so the answer is $\boxed{\frac{10}{9}\pi^4}$

5. $(3)(-1)(2)^2(1)^3 \cdot \frac{7!}{1!1!2!3!} = -\frac{(7!)(3)(2)(2)}{(2)(3)(2)} = -7! = \boxed{-5040}$

6. The question is equivalent to asking for the angle between $u = \langle 3,4 \rangle$ and $v = \langle 12,5 \rangle$. Using the formula $\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{36+20}{5 \cdot 13} = \frac{56}{65}$, so we get $\boxed{\cos\theta = \frac{56}{65}}$. Note, this question can also be solved by means of slopes and the tangent function (using a difference of angles formula) and then converting the value to cos.

7. $a = \frac{\log 3}{\log 5}$, for whichever base, so $\frac{1}{a} = \frac{\log 5}{\log 3} = \log_3 5$. So now the problem can be rewritten as $3^{\log_3 5} = \boxed{5}$

8. $\cos^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{5\pi}{12}\right) = \cos^2\left(\frac{\pi}{12}\right) + \sin^2\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) = \cos^2\left(\frac{\pi}{12}\right) + \sin^2\left(\frac{\pi}{12}\right) = \boxed{1}$

9. In a quartic $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, the sum of the roots taken two at a time is $\frac{c}{a} = \boxed{\frac{7}{4}}$. This can be seen by expanding $(x - r_1)(x - r_2)(x - r_3)(x - r_4)$ and collecting terms with same powers of x.

10. Let $u = 2^x$, then $u^2 = \left(2^x\right)^2 = 2^{2x} = \left(2^2\right)^x = 4^x$ and our equation becomes $u^2 - 5u + 6 = 0$, which has solutions $u = 2$ and $u = 3$. So replacing u with 2^x we get $2^x = 2$ and $2^x = 3$. Which solving for x yields $x = 1$ and $\boxed{x = \log_2 3} \approx 1.58$.

11. $\sum_{k=1}^{23}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{22}-\frac{1}{23}\right)+\left(\frac{1}{23}-\frac{1}{24}\right)=$

$\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\cdots+\frac{1}{22}\right)+\left(-\frac{1}{23}+\frac{1}{23}\right)-\frac{1}{24}=1+0+0+\cdots+0-\frac{1}{24}=1-\frac{1}{24}=\boxed{\frac{23}{24}}$

12. $(1-i)^7=(1-i)\cdot\left[(1-i)^2\right]^3=(1-i)\cdot\left(1-2i+i^2\right)^3=$

$(1-i)\cdot(-2i)^3=(1-i)\cdot\left(-8i^3\right)=\boxed{8i+8}$

13. 7 letters total, E is repeated 3 times, T is repeated 2 times, A and R occur once each. Therefore the total number of distinct permutations is $\frac{7!}{3!2!}=\frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot2}=7\cdot5\cdot4\cdot3=20\cdot21=\boxed{420}$

14. $f(1)=1^3-6(1)^2+11(1)-6=0$ so 1 is a root. Using synthetic division (or any similar method) we find that $f(x)=x^3-6x^2+11x-6=(x-1)\left(x^2-5x+6\right)$. The quadratic term here has solutions 2 and 3, so $100(1)+10(2)+(3)=\boxed{123}$.

15. $\det(B)=\det\left(A^4\right)=\det(A)\cdot\det(A)\cdot\det(A)\cdot\det(A)=\left[\det(A)\right]^4=(-2)^4=\boxed{16}$

16. The asymptotes of a hyperbola in this form will pass through the center of the hyperbola $(2,1)$ and have slopes $\pm\frac{\sqrt{9}}{\sqrt{4}}=\pm\frac{3}{2}$. We want the line with positive slope, so combining the point and slope we get $y-1=\frac{3}{2}(x-2)\quad\Rightarrow\quad y=\frac{3}{2}x-3+1=\ \frac{3}{2}x-2=y$. Which has y-intercept $\boxed{-2}$

17. $3x^2+y^2+12x+6y=15\quad\Rightarrow\quad 3\left(x^2+4x+4\right)+\left(y^2+6y+9\right)=15+12+9\quad\Rightarrow$

$3(x+2)^2+(y+3)^2=36\quad\Rightarrow\quad\frac{(x+2)^2}{12}+\frac{(y+3)^2}{36}=1$

$A=\pi\sqrt{12\cdot36}=6\cdot2\sqrt{3}\cdot\pi=\boxed{12\pi\sqrt{3}}$

18.
$$\begin{aligned}\sum_{n=1}^{\infty}n\cdot2^{-n}=\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots=\ &\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1+\frac{1}{2}+\frac{1}{4}+\cdots=\boxed{2}\\ +\ &\phantom{\frac{1}{2}+{}}\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\\ +\ &\phantom{\frac{1}{2}+\frac{1}{4}+{}}\frac{1}{8}+\frac{1}{16}+\cdots\\ &\vdots\qquad\qquad\quad\ddots\end{aligned}$$

19. Let $a=\overline{BC}$ and similarly for the other sides of the triangle, then using the Law of Cosines we get: $b=\sqrt{a^2+c^2-2ac\cos(B)}$, plugging in our given values this becomes,

$b=\sqrt{5^2+4^2-2\cdot5\cdot4\cdot\frac{1}{8}}=\sqrt{25+16-5}=\sqrt{36}=\ \boxed{6}$

20. The matrix given is in upper triangular form, so the determinant is simply the product along the diagonals. To see why this is so, evaluate the determinant using expansion by minors, first along the first column, then along the new first column, etc. So the answer is $1\cdot2\cdot3\cdot4=\boxed{24}$

21. $\left|x^2-1\right|\le\left|x^2+2x+1\right|\quad\Rightarrow\quad\left|(x-1)(x+1)\right|\le\left|(x+1)(x+1)\right|\quad\Rightarrow\quad\left|(x-1)\right|\le\left|(x+1)\right|\quad\Rightarrow\quad x\ge0$ but we can't forget about when $x+1=0$ i.e. $x=-1$ because in that case we couldn't have divided by $x+1$ to simplify our inequality. When $x=-1$ we get $0\le0$ which is true. So all solutions are $\boxed{x=-1\text{ and }0\le x}$.

22. The answer is given by $\frac{(5-1)!}{2} = \frac{24}{2} = \boxed{12}$. We subtract 1 from 5 before taking the factorial because (like seats around a table) there is no specific starting point. We divide by 2 because we can turn the keychain around (this would be equivalent to flipping over the table).

23. Using the distance formula from a plane $(Ax + By + Cz + D = 0)$ to a point (x_0, y_0, z_0), $dist = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ and substituting our plane and point, we get $dist = \frac{|1+2+3+4|}{\sqrt{1+4+9}} = \frac{10}{\sqrt{14}} = \frac{10\sqrt{14}}{14} = \boxed{\frac{5\sqrt{14}}{7}}$

24. The chances of rolling an odd number (o) for either die independently is $1/2$, similarly for an even number (e) with just one die is $1/2$. Now when adding two numbers together $e + o = o + e = o$, the chances of the first die being even when the second is odd is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, which is the same chances as the first die being odd when the second is even, so we add $\frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$.

25. $u \times v$ is perpendicular to u by definition of cross product. The dot product of two perpendicular vectors is always $\boxed{0}$.