Advanced Calculus Solutions
2007 Mu Alpha Theta National Convention

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1. $\int_0^{\pi/3} \cos^2 x dx = \frac{1}{2}\int_0^{\pi/3}(1+\cos 2x)dx = \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/3} = \frac{1}{2}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}-0\right) = \frac{4\pi+3\sqrt{3}}{24}$ **A**

2. $A = \frac{3}{2}\int_{-\pi/6}^{\pi/6}\cos^2 3\theta\, d\theta = \frac{3}{4}\int_{-\pi/6}^{\pi/6}(1+\cos 6\theta)\, d\theta = \frac{3}{4}\left[\theta+\frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6} = \frac{3}{4}\left(\frac{\pi}{3}+0\right) = \frac{\pi}{4}$ **C**

3. $\nabla f = \left(e^y + y^2\right)\hat{x} + \left(xe^y + 2xy\right)\hat{y}$ $\qquad \nabla f(2,1) = (e+1)\hat{x} + (2e+4)\hat{y}$ **A**

4. $SA = \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^2 x\sqrt{1+4x^2}dx = \frac{1}{8}\int_1^{17}\sqrt{u}du = \frac{1}{12}\left[u^{3/2}\right]_1^{17} = \frac{17\sqrt{17}-1}{12}$ **A**

5. $V = \iiint_V dV = \int_{-3}^{3}\int_0^{2\pi}\int_0^{\pi/6} r^2\sin\theta\, dr\, d\phi\, d\theta = \int_{-3}^{3} r^2 dr\int_0^{2\pi} d\phi \int_0^{\pi/6}\sin\theta\, d\theta = \left[\frac{r^3}{3}\right]_{-3}^{3}\left[\phi\right]_0^{2\pi}\left[-\cos\theta\right]_0^{\pi/6}$

   $... = (9-(-9))(2\pi-0)\left(1-\frac{\sqrt{3}}{2}\right) = 18\pi\left(2-\sqrt{3}\right)$ **A**

6. $\nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z} = x+0+\frac{xy}{2\sqrt{z}} = 2+\frac{2(1)}{2\sqrt{1}} = 3$ **B**

7. $\vec{f}'(t) = \cos t\hat{x} - \sin t\hat{y} + \hat{z}$ $\qquad \vec{f}''(t) = -\sin t\hat{x} - \cos t\hat{y}$ $\qquad \vec{f}''(-\pi/4) = \frac{\sqrt{2}}{2}\hat{x} - \frac{\sqrt{2}}{2}\hat{y}$ **A**

8. Centroid of ellipse: $(-4,6)$ — Distance from centroid to $y = x$: $5\sqrt{2}$
   Area of ellipse: $\pi(4)2 = 8\pi$ — Pappus: $V = 2\pi A r_c = 2\pi(8\pi)5\sqrt{2} = 80\pi^2\sqrt{2}$ **D**

9. $\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ...$ $\qquad \ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx .583$ **A**

10. Integrated by parts:

    $\int x^2\cos x dx = x^2\sin x - \int 2x\sin x du = x^2\sin x - 2\left(x(-\cos x) - \int -\cos x dx\right) = ...$

    $... = x^2\sin x + 2x\cos x - 2\sin x$

$$\int_{\pi/2}^{\pi} x^2 \cos x dx = \left[x^2 \sin x + 2x\cos x - 2\sin x\right]_{\pi/2}^{\pi} = \left(\pi^2(0) + 2\pi(-1) - 2(0)\right) - \left(\frac{\pi^2}{4}(1) + 2\frac{\pi}{2}(0) - 2(1)\right)$$

continued…

$$... = -\frac{\pi^2}{4} - 2\pi + 2 = \frac{-\pi^2 - 8\pi + 8}{4} \quad \textbf{E}$$

11. $\iint_A f(x,y)dA = \int_0^1\int_0^{1-x} x^2 y dy dx = \int_0^1 x^2\left[y^2/2\right]_0^{1-x} dx = \frac{1}{2}\int_0^1 x^2(1-x)^2 dx = \frac{1}{2}\int_0^1\left(x^2 - 2x^3 + x^4\right)dx = ...$

$$... = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{60} \qquad \textbf{A}$$

12. Area of a quarter circle of radius 1: $\frac{\pi}{4}$ (can also be solved substituting $x = \sin\theta$) **A**

13. $f'(t) = \hat{x} + 2t\hat{y} - \sin t\hat{z}$ $\quad f'(\pi/2) = \hat{x} + \pi\hat{y} - \hat{z}$ $\quad \left|f'(\pi/2)\right| = \sqrt{1+\pi^2+1}$ **D**

14. $\lim\limits_{x\to\infty}\frac{x\log_a x}{x\log_b x} = \lim\limits_{x\to\infty}\frac{x\log_a x}{x}\frac{\log_a b}{\log_a x} = \log_a b$ **B**

15. Domain: $\sqrt{x} = x \Rightarrow x = 0,1$

$$a = \frac{\int_0^1 x\left(\sqrt{x}-x\right)dx}{\int_0^1\left(\sqrt{x}-x\right)dx} = \frac{2}{5} \qquad b = \frac{\int_0^1\left(\sqrt{x}-x\right)\left(\sqrt{x}+x\right)dx}{2\int_0^1\left(\sqrt{x}-x\right)dx} = \frac{1}{2} \qquad \frac{a}{b} = \frac{1}{5} \quad \textbf{A}$$

16. $\lim\limits_{(x,y)\to(2,1)}\frac{xy}{xy^3 + xy} = \frac{2}{2+2} = \frac{1}{2}$ **C**

17. $\int_0^L\int_0^z\int_0^2 xy\cos z^3 dxdydz = \int_0^2 xdx\int_0^L \cos z^3\int_0^z ydydz = 2\int_0^L \cos z^3\left[\frac{y^2}{2}\right]_0^z dz = \int_0^L z^2\cos z^3 dz = ...$

$$... = \frac{1}{3}\left[\sin z^3\right]_0^L = \frac{1}{3}\sin(\pi/3) = \frac{\sqrt{3}}{6} \qquad \textbf{A}$$

18. Integration by parts: $\int \ln x dx = x\ln x - \int x/x dx = x\ln x - x$

$\int_1^2 \ln x dx = 2\ln 2 - 2 - 1\ln 1 + 1 = 2\ln 2 - 1$ **D**

19. $d = \int_0^{\sqrt{5}/3} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_0^{\sqrt{5}/3} \sqrt{4t^2 + 9t^4}\,dt = \int_0^{\sqrt{5}/3} t\sqrt{4 + 9t^2}\,dt = \frac{1}{18}\int_0^{\sqrt{5}/3} 18t\sqrt{4 + 9t^2}\,dt = \ldots$

$\ldots = \frac{1}{18}\frac{2}{3}\left[\left(4 + 9t^2\right)^{3/2}\right]_0^{\sqrt{5}/3} = \frac{1}{27}(27 - 8) = \frac{19}{27}$ **B**

20. $\int_0^1\int_0^{z^2}\int_1^{2^y} \frac{z^3 e^{y^2}}{x}\,dxdydz = \int_0^1 z^3 \int_0^{z^2} e^{y^2} \int_1^{2^y} \frac{dxdydz}{x} = \int_0^1 z^3 \int_0^{z^2} e^{y^2} y\ln 2 dydz = \frac{\ln 2}{2}\int_0^1 z^3 \int_0^{z^2} 2ye^{y^2}dydz = \ldots$

$\ldots = \frac{\ln 2}{2}\int_0^1 z^3\left(e^{z^4} - 1\right)dz = \frac{\ln 2}{8}\left[e^{z^4} - z^4\right]_0^1 = \frac{\ln 2}{8}(e - 1 - 1) = \frac{(e-2)\ln 2}{8}$ **E**

21. $f_x = 2(x-1)$ $\quad f_y = 2y$ $\quad \nabla f = 2(x-1)\hat{x} + 2y\hat{y} = 0 \Rightarrow (x,y) = (1,0)$ local

Boundary: $y = \pm\sqrt{4 - x^2}$ $\quad g(x) = f\left(x, \pm\sqrt{4 - x^2}\right) = (x-1)^2 + 4 - x^2 + 1 = -2x + 6$

$g'(x) = -2 \Rightarrow$ no local extrema on boundary curves, just check endpoints $(-2,0)$ $(0,2)$

$f(-2,0) = 10$ $\quad f(2,0) = 2$ $\quad f(1,0) = 1$ $\quad$ max: 10 $\quad$ min: 1 $\quad 10 - 1 = 9$ **D**

22. Area: 9 $\quad$ Centroid: $(5,5)$ $\quad d =$ distance from $(5,5)$ to $y = -3x + b$

Pappus: $\quad V = 2\pi(9)d = 18\pi d = 72\pi \Rightarrow d = 4$

$9 \pm 4 = 5, 13$ $\quad 5 + 13 = 18$ **C**

23. $\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 (1-u)\sqrt{u}(-du) = \int_0^1 \left(\sqrt{u} - u\sqrt{u}\right)du = \left[\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$ **B**

24. Cylindrical description of cylinder: $\quad r \in [0,2]$ $\theta \in [0,2\pi]$ $z \in [0,4]$

$x^2 + y^2 = r^2$ $\quad dV = rdrdzd\theta$

$m = \iiint_V \rho(x,y,z)dV = \int_0^{2\pi}\int_0^2\int_0^4 r^2 zrdzdrd\theta = \int_0^{2\pi} d\theta \int_0^2 r^3 dr \int_0^4 zdz = 2\pi(4)8 = 64\pi$ **B**

25. $x = t \Rightarrow y = t^2$ $\quad t \in [0,3]$

$\int_C f(x,y)ds = \int_0^3 \frac{t^2}{t}\sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_0^3 t\sqrt{4t^2 + 1} = \frac{1}{8}\frac{2}{3}\left[\sqrt{4t^2 + 1}\right]_0^3 = \frac{37\sqrt{37} - 1}{12}$ **B**

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26. $A = \int_0^{2\pi} [1+\cos\theta]^2 \, d\theta = \int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta = \frac{1}{2}\int_0^{2\pi} \left(2 + 4\cos\theta + 1 + \cos 2\theta\right) d\theta = \ldots$

$\ldots = \frac{1}{2}\left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 3\pi$ **C**

27. $\frac{\partial z}{\partial y} = x\left(2y\sin xy + xy^2 \cos xy\right) = \frac{\pi}{3}\left(2(2)\sin\frac{2\pi}{3} + \frac{\pi}{3}(4)\cos\frac{2\pi}{3}\right) = \frac{\pi}{3}\left(2\sqrt{3} - \frac{2\pi}{3}\right) = \frac{2\pi\left(3\sqrt{3} - \pi\right)}{3}$ **D**

28. $x'(t) = 2te^{t^2}$ $\quad x'(2) = 4e^4$ $\quad \tan\theta = dy/dx = \frac{6e^6}{4e^4} = \frac{3}{2}e^2$ **C**

$y'(t) = 6e^{3t}$ $\quad y'(2) = 6e^6$

29. $f(x) = \frac{(x^2-4)(x^2-x-12)}{(x^2-5x+6)(x+4)} = \frac{(x-2)(x+2)(x-4)(x+3)}{(x-2)(x-3)(x+4)}$

Two asymptotes for poles that aren't cancelled by roots (vertical asymptotes), and one because the numerator is 1 degree higher than the denominator (slant asymptote). **C**

30. $\frac{\partial V}{\partial r} = 2\pi rh = 2\pi(2)3 = 12\pi$ **C**