# Mu Alpha Theta National Convention: 2007
## Solutions – Ciphering Test – Mu Division

0. How many positive even integer solutions exist for the equation $a+b+c+d=100$?

   **Answer:** 18424

   **Solution:** Since *a*, *b*, *c*, and *d* are even integers, this corresponds to having 50 2's in a row, then placing 3 0's between them. For example, (16, 24, 36, 24) would correspond to

   22222222022222222222220222222222222222222022222222222.

   Since there are 49 spaces to put the 3 zeroes, there are $\binom{49}{3}=18424$ even integer solutions to the given equation.

1. A hemispherical bowl of radius 5 units is filled with water to a height of 2 units. How much more volume is needed to completely fill the bowl?

   **Answer:** $66\pi$

   **Solution:** Using washer method, take the bowl to be $y=\sqrt{25-x^2}$ revolved about the *x*-axis from 0 to 5. To fill to the rest of the 3 units of height, the integral is evaluated from 0 to 3:

   $$\pi\int_0^3\left(\sqrt{25-x^2}\right)^2dx=\pi\int_0^3\left(25-x^2\right)dx=\pi\cdot\left(25x-\frac{x^3}{3}\Big|_0^3\right)=66\pi$$

2. Let $f(x)=x^x$. Find the sum of all values of *x* for which $f(x)=f'(x)$.

   **Answer:** 1

   **Solution:** $f(x)=x^x\to\ln f(x)=x\ln x$. Differentiating, we have $\frac{f'(x)}{f(x)}=1+\ln x$.

   Rearranging $f(x)=f'(x)$ into $\frac{f'(x)}{f(x)}=1$, we see the values of *x* are found by solving $1=1+\ln x\to\ln x=0$ which yields the only solution $x=1$.

3. If $f(x)=x^3+ax^2+bx+c$, find the product $abc$ given that $f(x)$ has critical points at $x=-1$ and $x=5$, and that $f(-1)=9$.

   **Answer:** 90

   **Solution 1:** $f'(x)=3x^2+2ax+b$ has critical points at $x=-1$ and $x=5$. This implies $f'(-1)=3-2a+b=0$ and $f'(5)=75+10a+b=0$. Solving the $2\times 2$ system of equations

   $$\begin{array}{l}-2a+b=-3\\10a+b=-75\end{array}\to 12a=-72\to a=-6\to b=-15.$$

   Since *a* and *b* are found, $f(x)=x^3-6x^2-15x+c$. Given the initial condition $f(-1)=9$, $f(-1)=-1-6+15+c=9\to c=1$. Therefore $abc=(-6)(-15)(1)=90$.

**Solution 2:** Since $f(x)$ has critical points at $x=-1$ and $x=5$, $f'(x)=k(x+1)(x-5)$, $k$ constant. Since the leading coefficient of $f(x)$ is 1, then the coefficient of $x^2$ in $f'(x)$ must be eliminated in the process of antidifferentiation. This means $\int kx^2 = x^3 \to \frac{1}{3}k=1 \to k=3$, which implies $f'(x)=3(x+1)(x-5)=3x^2-12x-15$ and $f(x)=x^3-6x^2-15x+c$. Then using $f(-1)=9$, $f(-1)=-1-6+15+c=9 \to c=1$. Therefore $abc=(-6)(-15)(1)=90$.

4. If $f(x)$ is a differentiable and concave down quadratic polynomial on the interval [0,10], and if $f(x)<0$ on [0,10] with a relative maximum at $x=2$, put the letters representing these approximations in order from smallest to largest.

$A - \int_0^{10} f(x)dx$
*B* – Left Hand Approximation of *A* using 10 rectangles of equal base length.
*C* – Right Hand Approximation of *A* using 10 rectangles of equal base length.
*D* – Midpoint Approximation of *A* using 10 rectangles of equal base length.
*E* – Trapezoid Approximation of *A* using 10 intervals of equal base length.
**Answer:** *CEADB*
**Solution:** $f(x)$ is concave down and negative for all values on its interval. This means that left hand approximation will always over-approximate (since the definite integral is negative, this means less negative) the value of the definite integral and right hand rule will always under-approximate the value. Trapezoid approximation will always under-approximate, but it will be greater than the right hand approximation, and the value of the definite integral will be greater than the trapezoid approximation. The midpoint approximation will over approximate the area because the area of the under-approximation is greater (less negative) than the over-approximation left out which is calculated by the definite integral.
*Note:* These results can be verified by examining a quadratic satisfying the given conditions.

5. Let ***A*** be the volume of the solid formed by rotating $y=4-x$ around the *x*-axis on the interval $x=0$ to $x=2$. Let ***B*** be the volume of the solid formed by the region bound by $x=2$, $y=4-x$, and $y=k$, where $k>4$. For what exact value of *k* is ***A*** = ***B***?

**Answer:** $\dfrac{2\sqrt{42}}{3}$

**Solution:** $A=\pi\int_0^2 (4-x)^2\,dx$ and $B=\pi\int_0^2 \left[k^2-(4-x)^2\right]dx$. Setting these two equal yields

$$\pi\int_0^2 (4-x)^2 = \pi\int_0^2 k^2 dx - \pi\int_0^2 (4-x)^2\,dx$$
$$2\int_0^2 (4-x)^2 = \int_0^2 k^2 dx$$
$$2k^2 = 2\cdot\left.\frac{-(4-x)^3}{3}\right|_0^2$$

$$k^2=\frac{-8}{3}+\frac{64}{3}=\frac{56}{3}$$
$$k=\sqrt{\frac{56}{3}}=\frac{2\sqrt{42}}{3}$$

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6. A sphere of radius $\pi/2$ has a volume charge density of $p(r)=\dfrac{\sin(r)}{r}$. What is the total charge ($Q$) enclosed in the sphere? (Hint: $dQ = p \cdot dV$ )

**Answer:** $2\pi^2 + 4\pi$

**Solution:**
$dV = 4\pi r^2 dr$
$dQ = p(r)dV = (\sin(r)4\pi r^2 dr)/r$
$dQ = 4\pi r\sin(r)dr$
$Q = \int(4\pi r\sin(r)dr, r, 0, \pi/2)$ = integration by parts
$= 4\pi ( -r*\cos(r) + \sin(r) |0$ and $\pi/2)$
$= 2\pi^2 + 4\pi$

7. An ellipse with a horizontal major axis of 6 and a vertical minor axis of 4 is revolved about its horizontal axis to form an ellipsoid. At time t = 0, a plane begins to pass through the ellipse perpendicular to its horizontal axis at a rate of 1 unit/sec. When t = 4 sec, at what rate is the cross-sectional area of the plane changing?

**Answer:** $-8\pi/9$

**Solution:**
$x^2/9 + y^2/4 = 1$.
When t = 4 seconds, x = 1.
Area = A = $\pi y^2$ (cross sectional area is a circle) = $\pi(36-4x^2)/9$
$dA/dt = \pi(-8x)/9*dx/dt$, $dx/dt = 1$ and $x = 1$,
$= -8\pi/9$

8. Given $f(x)=x^3-2x-4$ and $f^{-1}(x)=g(x)$ what is $g'(0)$?

**Answer:** $\dfrac{1}{10}$

**Solution:** $f(g(x))=x \to f'(g(x))g'(x)=1 \to g'(x)=\dfrac{1}{f'(g(x))}$ so $g'(0)=\dfrac{1}{f'(g(0))}$.

$g(0)=2$, and $f'(2)=10$, so $g'(0)=\dfrac{1}{10}$.

9. Let

$A =$ $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

$B =$ The area of a regular $n$-gon whose distance from the center to a vertex is 1 unit as $n \to \infty$.

What is $A^{2B}$ (simplified)?

**Answer:** $\pi^{\pi}$

**Solution:** $A = \sqrt{\pi}$ and $B$ interpretively is a circle with radius 1, whose area is $\pi$. $A^{2B} = \left(\sqrt{\pi}\right)^{2\pi} = \pi^{\pi}$.

10. Evaluate: $\int_{-1}^{1} \frac{7x^{316}\sin\left(x^{325}\right) + 2x^{110} + x^{332}}{1+x^{222}}\,dx$

**Answer:** $\frac{\pi+4}{222}$ or equivalent

**Solution:** The function $y = \frac{7x^{316}\cdot\sin x^{325}}{1+x^{222}}$ is odd, which means it is symmetric with respect to the origin, so $\int_{-1}^{1}\frac{7x^{316}\sin x^{325}}{1+x^{222}}dx = 0$. This leaves $\int_{-1}^{1}\frac{2x^{110}+x^{332}}{1+x^{222}}dx$, to which we substitute $u = x^{111}$ to get $du = 111x^{110}dx \to dx = \frac{1}{111u^{110/111}}du$. So

$$\int_{-1}^{1}\frac{2x^{110}+x^{332}}{1+x^{222}}dx = \int_{-1}^{1}\frac{2\left(u^{1/111}\right)^{110}+\left(u^{1/111}\right)^{332}}{1+\left(u^{1/111}\right)^{222}}\cdot\frac{1}{111u^{110/111}}du = \frac{1}{111}\int_{-1}^{1}\left(\frac{2+u^2}{1+u^2}\right)du$$

$$\frac{2}{111}\int_{0}^{1}\left(\frac{2+u^2}{1+u^2}\right)du = \frac{2}{111}\int_{0}^{1}\left(1+\frac{1}{1+u^2}\right)du = \frac{2}{111}\left(x+\arctan x\right)\Big|_0^1 = \frac{2}{111}\left(1+\frac{\pi}{4}\right) = \frac{4+\pi}{222}.$$

10. Evaluate: $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{k^2+n^2}$

**Answer:** $\frac{\pi}{4}$

**Solution:** $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{k^2+n^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{n^2}{k^2+n^2}\right)\cdot\frac{1}{n}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{(k/n)^2+1}\right)\cdot\frac{1}{n}$

$=\int_0^1\frac{1}{x^2+1}dx=(\arctan x)\big|_0^1=\frac{\pi}{4}.$