2007 Mu Alpha Theta National Convention
Mu Limits and Derivatives-SOLUTIONS

B 1.) $f'(x) = 3x^2 + 1$. $f'(3) = 27 + 1$. $f'(3) = 28$.

C 2.) $\frac{df}{df'} = \frac{\frac{df}{dx}}{\frac{df'}{dx}} = \frac{f'(x)}{f''(x)}$. $f'(2) = 13, f''(2) = 12$. Thus $\frac{df}{df'} = \frac{13}{12}$.

D 3.) $\lim_{t\to-\infty} f(t) = 1$. $\lim_{t\to\infty} f(t) = 1$. $\lim_{t\to3} f(t) = \lim_{x\to3} \frac{x^2+1}{x^2+2} = \frac{10}{11}$. Thus $22 + 22 + 20 = 64$.

D 4.) By L'Hopital's Rule: $\lim_{h\to0} \frac{f(x+h)^2 - f(x)^2}{2h} = \lim_{h\to0} \frac{2f(x+h)f'(x+h)}{2} = f(x)f'(x)$. Thus, $f(1)f'(1) = (1+1)(3) = 6$.

B 5.) $A = \frac{1}{2}AB\sin c$, $A' = \frac{1}{2}ABc'\cos c = 8\sqrt{2}c'\cos c$. Since $a + b + c = \pi$, $c = \frac{7\pi}{12}$. Since $a' + b' + c' = 0$, $c' = \frac{\pi}{12}$. Thus, $A' = 8\sqrt{2}\frac{\pi}{12}\cos\frac{7\pi}{12} = -\frac{2\sqrt{2}\pi\cos\frac{5\pi}{12}}{3} = -\frac{2\sqrt{2}\pi\sin\frac{\pi}{12}}{3} = -\frac{\sqrt{2}(\sqrt{6}-\sqrt{2})\pi}{6} = \frac{(1-\sqrt{3})\pi}{3}$.

D 6.) Since $y = \ln x$ only exists on the domain $(0, \infty)$, there can be no tangent line drawn at $x = -4$.

A 7.) By L'Hopital's Rule: $\lim_{x\to3^+} \frac{\sqrt{x^2-9} - \sqrt{6x-18}}{x-3} = \lim_{x\to3^+} (\frac{x}{\sqrt{x^2-9}} - \frac{3}{\sqrt{6}\sqrt{x-3}}) = \lim_{x\to3^+} \frac{x - \frac{\sqrt{6}}{2}\sqrt{x+3}}{\sqrt{x^2-9}} = \lim_{x\to3^+} \frac{1 - \frac{\sqrt{6}}{4\sqrt{x+3}}}{\frac{x}{\sqrt{x^2-9}}} = \lim_{x\to3^+} \frac{(1 - \frac{\sqrt{6}}{4\sqrt{x+3}})\sqrt{x^2-9}}{x} = 0$.

D 8.) The last time the particle's y-coordinate is 18 if when $x = 100$. Since $y' = (200\sin 100\pi + 10000\pi\cos 100\pi + \frac{1}{100\ln 10})x'$, $y' = 20000\pi + \frac{2}{100\ln 10}$. Thus, 20000π is closest to y'.

B 9.) $A = \pi r^2 = 4\pi$. Thus, $r = 2$. $A' = 2\pi rr' = 2\pi$. Thus, $r' = \frac{1}{2}$. If B is the area inside the square, but outside of the circle, then $B = r^2 - \frac{\pi r^2}{4}$. $B' = 2rr' - \frac{\pi rr'}{2} = 2 - \frac{\pi}{2}$.

B 10.) $\frac{dx}{dy} = 2y + 2$. When $x = 1, y = -1$. Thus, $\frac{dx}{dy} = 0$.

B 11.) The probability of rolling a one in any one roll is $\frac{1}{n}$. The probability of not rolling a one, in n consecutive rolls, is $(1-\frac{1}{n})^n$. Thus the probability of rolling a one at least once, as n approaches infinity, is $\lim_{x\to\infty}(1-(1-\frac{1}{n})^n) = 1-\frac{1}{e}$. Since $\frac{1}{2} > \frac{1}{e} > \frac{1}{4}$, this probability is closest to $\frac{1}{2}$.

C 12.) The additional profit due to the stars is given by $(f(x) + g(x)) - f(x) = g(x)$. Thus, the difference in marginal profit is $(f'(20) + g'(20)) - f'(20) = g'(20) = -1200$. The absolute value of this is 1200.

E 13.) The angle between the two tangent lines, which will have slopes of 5 and $3a^2 - 7$, must be 45 degrees. Thus, $\tan\frac{\pi}{4} = \frac{12-3a^2}{15a^2-34}$. $12 - 3a^2 = 15a^2 - 34$. $a^2 = \frac{23}{9}$. $a = \frac{\sqrt{23}}{3}$, $9 \cdot a = 3\sqrt{23}$.

B 14.) $\lim_{n\to\infty} \sum_{i=0}^{n} \frac{ni - i^2}{n^3} = \lim_{n\to\infty} \sum_{i=0}^{n} (\frac{i}{n^2} - \frac{i^2}{n^3}) = \lim_{n\to\infty} (\frac{\frac{n(n+1)}{2}}{n^2} - \frac{\frac{n(n+1)(2n+1)}{6}}{n^3}) = (\frac{1}{2} - \frac{1}{3}) = \frac{1}{6}$.

A 15.) $p(t)$ will be differentiable for all $t > 0$ if $\lim_{t\to2^-} v(t) = \lim_{t\to2^+} v(t)$. Thus, $4 = 4k + 3$, and $k = \frac{1}{4}$.

A 16.) The area of such an ellipse is $ab\pi$. By Pappus's Theorem, if such a shape is rotated about a line c units away from it's centroid, the resulting object's volume is $(2c\pi)(ab\pi) = 2abc\pi^2$. There is no need to maximize anything, there is only one such shape.

B 17.) $g'(c) = \frac{0-(-1)}{1-0} = 1$. $\frac{c}{\sqrt{1-c^2}} = 1$. $c^2 = 1 - c^2$. $c = \frac{\sqrt{2}}{2}$.

C 18.) Since $g'(x)$ must equal zero, the only x-value which satisfies Rolle's Theorem is $x = 0$.

E 19.) The tangent lines at $x = \frac{3}{2}$ will have slopes -1 and 1 since $f(x) = x$ and $f(x) = -x$ are their own inverses. For the tangent lines at $x = -\frac{1}{2}$, we can say that if $h(x)$ is the inverse of $g(x)$, then $g(h(x)) = x$, thus $g'(h(-\frac{1}{2}))h'(-\frac{1}{2}) = 1$, thus $h'(-\frac{1}{2}) = \frac{1}{g'(\pm\frac{\sqrt{3}}{2})} = \pm\frac{\sqrt{3}}{3}$. Thus the sum of the absolute values of the slopes is $2 + \frac{2\sqrt{3}}{3}$.

A 20.) $f'(k) = \frac{f(k)+2}{k+2}$. $(2k+2)(k+2) = k^2 + 2k + 5$. $k^2 + 4k - 1 = 0$. The sum of the roots is -4.

B 21.) Although the first step assumes that x is an integer, it is correct. The second step is faulty, since the right side in step 1 is not a continuous function, and thus the derivative cannot be taken.

A 22.) $\frac{d}{dx}\int_{\cos x}^{\sin x} t\,dt = \left(\frac{d}{dx}\int_0^{\sin x} t\,dt - \frac{d}{dx}\int_0^{\cos x} t\,dt\right) = (\cos x \sin x + \sin x \cos x) = \sin 2x$.

A 23.) $\frac{d}{dx}\int_{\sinh^2 x}^{\cosh^2 x} t\,dt = \left(\frac{d}{dx}\int_0^{\cosh^2 x} t\,dt - \frac{d}{dx}\int_0^{\sinh^2 x} t\,dt\right) = (2\cosh x \sinh x \cosh^2 x - 2\sinh x \cosh x \sinh^2 x) =$ $2\cosh x \sinh x(\cosh^2 x - \sinh^2 x) = \sinh 2x$.

C 24.) $f(x) = 2^x$. $c = 6$. $f'(6) = 64\ln 2$.

B 25.) $y = x^y$, so $\ln y = y \ln x$, $\frac{\ln y}{y} = \ln x$. We need to solve for k in the following: $\lim_{y\to k} \frac{\ln y}{y} = \lim_{x\to 0^+} \ln x$. Since the right diverges to $-\infty$, so must the left, and thus k must be 0^+. Thus, y, and the limit, approaches 0.

C 26.) As t approaches infinity, the c becomes irrelevant, so $\lim_{x\to\infty} \sum_{n=0}^{\infty} \frac{f(\frac{x}{2})^n}{f(x)^n} = \lim_{x\to\infty} \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^n}{x^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

E 27.) $\lim_{x\to-\infty} \frac{x + x^2 + x^3}{\sqrt{9x^6 + 4x^4 + x^2}} = \lim_{x\to-\infty} \frac{x}{3|x|} = -\frac{1}{3}$.

B 28.) First, we must find the perpendicular distance between I-80 and I-90. Since Gabor drove 180 miles on I-80, he drove $\frac{88}{19} - 2 = \frac{50}{19}$ hours on WYO-380. Since the speed limit here is 38, this section is 100 miles long. The time spent to reach I-90 will thus be $t(k) = \frac{k}{90} + \frac{100}{20+\frac{k}{10}}$, where k is the number of miles he drove on I-80 before turning onto a Wyoming road. $t'(k) = \frac{1}{90} - \frac{1000}{(200+k)^2}$. $t'(k) = 0$ when $(200+k)^2 = 90000$, thus $200 + k = 300$, $k = 100$. Thus, the time spent in reaching I-90 is minimized by driving for 100 miles and thus taking WYO-300.

C 29.) As long as Gabor's average speed, following the speed limit, is somewhere between 0 mph and 125 million mph, the closest answer will be 10.

B 30.) Gabor's function is $y = 0$. Since this is both even and odd, Gabor is always wrong and always right. Paul's function is the derivative of $y = 2ax^2$, so it is $y = 4ax$. Since this is odd for all $a \neq 0$, Paul is always wrong. Gus's and Praveen's functions are always neither even nor odd, thus they are both neither right nor wrong. Thus, B is false.