2007 Mu Alpha Theta National Convention
Calculus Matrices and Vectors Solutions

1. $\mathbf{u}\cdot\mathbf{v} = 2-3+0=-1$. **A**

2. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{vmatrix} = -5\mathbf{i}-5\mathbf{j}-5\mathbf{k}$. **E**

3. $\mathbf{u}\cdot\mathbf{v} = 3-\sqrt{3}+\sqrt{3}+1=4$. $\|\mathbf{u}\| = \sqrt{3+1}=2$. $\|\mathbf{v}\| = \sqrt{3-2\sqrt{3}+1+3+2\sqrt{3}+1} = 2\sqrt{2}$.
$\cos(\theta) = \dfrac{4}{2\cdot 2\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. $\theta = 45°$. **C**

4. $2A-3B = \begin{bmatrix} 3 & -10 \\ -11 & 8 \end{bmatrix}$. $\det(2A-3B) = 24-110=-86$. **E**

5. $\cos(\theta) = \dfrac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \dfrac{\sin(t)+\cos(t)}{\sqrt{2}}$. $-\sin(\theta)\,d\theta = \dfrac{\cos(t)-\sin(t)}{\sqrt{2}}dt$. $\dfrac{d\theta}{dt} = \dfrac{\sin(t)-\cos(t)}{\sin(\theta)\cdot\sqrt{2}}$.
$\sin(\theta) = \dfrac{\|\mathbf{x}\times\mathbf{y}\|}{\|\mathbf{x}\|\|\mathbf{y}\|}$. $\mathbf{x}\times\mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ \sin(t) & \cos(t) & 0 \end{vmatrix} = (\cos(t)-\sin(t))\mathbf{k}$. $\sin(\theta) = \dfrac{|\cos(t)-\sin(t)|}{\sqrt{2}}$ so
$\dfrac{d\theta}{dt} = \dfrac{\sin(t)-\cos(t)}{|\cos(t)-\sin(t)|} = 1$ when $t = \dfrac{\pi}{3}$. **A**

6. I is true by definition and II follows from I. Since the eigenvalues are the roots of the characteristic polynomial, and since the characteristic polynomial of an $n\times n$ matrix is an nth order polynomial, there are n roots by the Fundamental Theorem of Algebra. Since λ is an eigenvalue of A, there is a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Then $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$ and hence λ^2 is an eigenvlaue of A^2, so IV is true. **D**

7. A matrix is singular when its determinant is equal to 0. $\begin{vmatrix} x & 2 & 0 \\ -1 & 1 & 1 \\ x & 0 & 2 \end{vmatrix} = 4x+4=0$. $x=-1$. **A**

8. $\langle t, \cos(kt)\rangle = \displaystyle\int_{-\pi}^{\pi} t\cos(kt)\,dt$. Notice that $t\cos(kt)$ is an odd function, so the integral is 0 and hence $a_k = 0$. **B**

9. $\sin(\theta) = \dfrac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}$ and $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, so $\dfrac{\|\mathbf{u}\times\mathbf{v}\|^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} + \dfrac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} = 1$. Therefore
$(\mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2$. **C**

10. Adding a scalar multiple of one row to another row does not alter a matrix's determinant. Multiplying a row by a scalar multiple increases a matrix's determinant by a factor of the scalar. The determinant of the new matrix is therefore $12\cdot 5 = 60$. **C**

11. $\det(A) = \begin{vmatrix} 4x+1 & 5-2x \\ 2-2x & x-2 \end{vmatrix} = (4x+1)(x-2)-(5-2x)(2-2x) = 7x-12$.

$A^{-1} = \begin{bmatrix} \frac{x-2}{7x-12} & \frac{2-5x}{7x-12} \\ \frac{2x-2}{7x-12} & \frac{4x+1}{7x-12} \end{bmatrix}$. $\lim_{x\to\infty} A^{-1} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{4}{7} \end{bmatrix}$. The largest entry is $\frac{4}{7}$. **C**

12. Notice that after t seconds the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\frac{\pi}{4}+\frac{\pi}{32}t$. Geometrically, $\|\mathbf{v}_1-\mathbf{v}_2\|$ is the distance between the heads of the two vectors. That is, it is the distance between the points $(1,0)$ and $\left(\cos\left(\frac{\pi}{4}+\frac{\pi}{32}t\right),\sin\left(\frac{\pi}{4}+\frac{\pi}{32}t\right)\right)$. Then

$\|\mathbf{v}_1-\mathbf{v}_2\| = \sqrt{\left(1-\cos\left(\frac{\pi}{4}+\frac{\pi}{32}t\right)\right)^2+\sin^2\left(\frac{\pi}{4}+\frac{\pi}{32}t\right)} = \sqrt{2-2\cos\left(\frac{\pi}{4}+\frac{\pi}{32}t\right)}$.

$\frac{d\|\mathbf{v}_1-\mathbf{v}_2\|}{dt} = \frac{2\sin\left(\frac{\pi}{4}+\frac{\pi}{32}t\right)\cdot\frac{\pi}{32}}{2\sqrt{2-2\cos\left(\frac{\pi}{4}+\frac{\pi}{32}t\right)}}$. When $t=8$, $\frac{d\|\mathbf{v}_1-\mathbf{v}_2\|}{dt} = \frac{\pi\sqrt{2}}{64}$. **B**

13. I is a 3×3 matrix. II is a 2×2 matrix. III is a 3×3 matrix. IV is a 2×2 matrix. **D**

14. The area of R is given by $\int_0^1 \left(2-x^2-x^3\right)dx = 2x-\frac{x^3}{3}-\frac{x^4}{4}\Big|_0^1 = \frac{17}{12}$. The transformation matrix is $T = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$ and $\det T = 12$. The area of R' is the area of R multiplied by the determinant of the transformation matrix. $\frac{17}{12}\cdot 12 = 17$. **C**

15. The first equation implies $x=6-z$. Plugging into the second equation gives $y+z=6$. This combined with the first equation implies $x=y$ and the vector **v** will be of the form $\mathbf{v}=(t,t,6-t)$. $\|\mathbf{v}\| = \sqrt{3t^2-12t+36}$. We can minimize $3t^2-12t+36$ for simplicity. $6t-12=0$ so $t=2$ and hence $\|\mathbf{v}\| = \sqrt{4+4+16} = 2\sqrt{6}$. **B**

16. Notice that $y=x^4+3$ and $x\in\left[1,\sqrt{2}\right]$. $\int_1^{\sqrt{2}} \left(x^4+3\right)dx = \left[\frac{x^5}{5}+3x\right]_1^{\sqrt{2}} = \frac{19\sqrt{2}-16}{5}$. **E**

17. We want to find a vector (x,y,z) such that $x+z=0$ and $-x+2y+z=0$. Choose $x=1$. Then $z=-1$ and $y=1$. $(1,1,-1)$ is normal to both vectors. **D**

18. $|A_n| = \sqrt{n^2+2n}-\sqrt{n^2+2n+1} = \frac{-1}{\sqrt{n^2+2n}+\sqrt{n^2+2n+1}}$. $\lim_{n\to\infty}|A_n| = 0$. **A**

19. The area of the triangle is the absolute value of $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 4t-t^2 & t & 1 \\ t/2 & 2 & 1 \end{vmatrix} = \left|4t-\frac{5}{4}t^2\right|$. Since this does not change signs when $t \in [0,3]$, we need only maximize $4t-\frac{5}{4}t^2$. $4-\frac{5}{2}t=0$ so $t=\frac{8}{5}$. The area is then $4\cdot\frac{8}{5}-\frac{5}{4}\left(\frac{8}{5}\right)^2=\frac{16}{5}=3.2$. Checking the end points, the area would be 0 and .75, so the maximum is 3.2. **C**

20. Notice that, in general, the $b_{i,j}$ element of $B^T B$ is the dot product of the *i*th column of *B* and the *j*th column of *B*. Since $A^T A = I$, it follows that the magnitude of each column is 1 and that the columns are all orthogonal. Therefore I and II are both true. A counterexample for III is the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ whose determinant is –1. A counterexample for IV is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. **B**

21. $\begin{vmatrix} 1 & 0 & 5 \\ 0 & 4 & 1 \\ -2 & 3 & 2 \end{vmatrix} = 1\cdot 5 - 0 + 5\cdot 8 = 45$. **C**

22. $f'(x)=\left(2-\sqrt{3}\right)e^{\left(2-\sqrt{3}\right)x}$. $f'(0)=2-\sqrt{3}$. Two points of the tangent line are the origin and $\left(1,2-\sqrt{3}\right)$. If we rotate these two points $30°$, they will determine the rotated line. The rotation matrix is $\begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$. The origin will clearly be mapped to itself.

$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2-\sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3}-1 \\ \sqrt{3}-1 \end{bmatrix}$. The slope of the rotated line is 1. **B**

23. From the last equation, $5x-2=y+z$ so $x+y+z=6x-2$. Using Cramer's Rule,

$$x=\frac{\begin{vmatrix} -1 & 3 & 1 \\ 13 & 2 & 3 \\ 2 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 5 & -1 & -1 \end{vmatrix}} = \frac{-1\cdot 1-3\cdot -19+1\cdot -17}{2\cdot 1-3\cdot -16+1\cdot -11} = \frac{39}{39} = 1.$$ $x+y+z=4$. **C**

24. The speed is given by $\left\|\mathbf{r}'(0)\right\|$. $\mathbf{r}'(t)=(6t+2)\mathbf{i}+\left(\cos(t)-t\sin(t)\right)\mathbf{j}+e^t\mathbf{k}$. $\mathbf{r}'(0)=2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\left\|\mathbf{r}'(0)\right\|=\sqrt{6}$. **D**

25. Since $\det A = \frac{1}{6}$, $\lim_{n\to\infty}\sum_{i=1}^{n}\det\left(A^i\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\det A\right)^i = \sum_{i=1}^{\infty}\left(\frac{1}{6}\right)^i = \frac{\frac{1}{6}}{1-\frac{1}{5}} = \frac{1}{5}$. **B**

26. M is row equivalent to the identity matrix, hence the rank of A is 4. **E**

27. It is easy to see that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. Using a Maclaurin expansion,

$$\cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \cdots = \begin{bmatrix} 1-\frac{1}{2!}+\frac{1}{4!}-\cdots & 0-\frac{2}{2!}+\frac{4}{4!}-\cdots \\ 0 & 1-\frac{1}{2!}+\frac{1}{4!}-\cdots \end{bmatrix} = \begin{bmatrix} \cos(1) & -\sin(1) \\ 0 & \cos(1) \end{bmatrix}.$$ **A**

28. $\begin{vmatrix} 2-3a & 3a+1 \\ a-2 & 1-a \end{vmatrix} = (2-3a)(1-a)-(3a+1)(a-2) = 3a^2-5a+2-\left(3a^2-5a-2\right) = 4$. **B**

29. $(-1,3)\oplus(5,0) = \left\|(4,3)\right\| - [-5+0] = 5+5 = 10$. **D**

30. $\frac{2\cdot 1 - 1\cdot 2 + 2\cdot 6 + 3}{\sqrt{2^2+(-1)^2+2^2}} = \frac{15}{3} = 5$. **C**