Mu Matrices and Vectors
2007 Mu Alpha Theta National Convention
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For all questions, answer E. "NOTA" means none of the above answers is correct. Unless otherwise stated, assume all numbers are real.

For Problems 1 and 2, let $\mathbf{u} = (2, 3, -5)$ and $\mathbf{v} = (1, -1, 0)$.

1. Evaluate $\mathbf{u} \cdot \mathbf{v}$.

   A. -1 B. 0 C. 2 D. 5 E. NOTA

2. Evaluate $\mathbf{u} \times \mathbf{v}$.

   A. $\mathbf{0}$ B. $\mathbf{i} - \mathbf{j}$ C. $\mathbf{i} + \mathbf{j} + \mathbf{k}$ D. $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ E. NOTA

3. What is the measure of the acute angle between the vectors $\mathbf{u} = \left(\sqrt{3}, 1\right)$ and $\mathbf{v} = \left(\sqrt{3} - 1, \sqrt{3} + 1\right)$?

   A. 15° B. 30° C. 45° D. 60° E. NOTA

4. Let $A = \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$. What is the value of $\det(2A - 3B)$?

   A. -12 B. 56 C. 106 D. 142 E. NOTA

5. Let $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (\cos(t), \sin(t))$ be vectors where $t \geq 0$. At what rate is the acute angle between $\mathbf{x}$ and $\mathbf{y}$ changing when $t = \frac{\pi}{3}$?

   A. 1 B. $\frac{\pi}{12}$ C. $\frac{1}{2}\left(\sqrt{3} - \sqrt{2}\right)$ D. $\frac{1}{6}\left(3\sqrt{2} - 2\sqrt{3}\right)$ E. NOTA

6. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Which of the following statements are true?

   I. $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \in$ ¡ ^{n}
   II. $\det(A - \lambda I) = 0$
   III. A has n complex eigenvalues, counting multiplicity
   IV. λ^2 is an eigenvalue of A^2

   A. I and II only B. II, III, and IV only
   C. I, II, and III only D. I, II, III, and IV E. NOTA

7. For what value of x is the matrix $\begin{bmatrix} x & 2 & 0 \\ -1 & 1 & 1 \\ x & 0 & 2 \end{bmatrix}$ singular?

A. -1 B. 0 C. $\frac{1}{2}$ D. 1 E. NOTA

8. Define the inner product $\langle f(t), g(t) \rangle$ of two continuous functions f and g on $[-\pi, \pi]$ by $\langle f(t), g(t) \rangle = \int_{-\pi}^{\pi} f(t) \cdot g(t)\, dt$. Evaluate a_k if $a_k = \frac{\langle t, \cos(kt) \rangle}{\langle \cos(kt), \cos(kt) \rangle}$.

A. $-\frac{2}{k}$ B. 0 C. $\frac{2}{k}$ D. $\frac{k}{\pi}$ E. NOTA

9. Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^3$. Which of the following is equal to $(\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2$?

A. $\|\mathbf{u}\|\|\mathbf{v}\|$ B. $\|\mathbf{u}\| + \|\mathbf{v}\|$ C. $\|\mathbf{u}\|^2 \|\mathbf{v}\|^2$ D. $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ E. NOTA

10. Let M be a 5×5 matrix such that $\det(M) = 12$. The matrix M' is formed by replacing the third row of M by the five times the third row of M minus two times the second row of M. What is $\det(M')$?

A. 12 B. 36 C. 60 D. 96 E. NOTA

11. Let $A = \begin{bmatrix} 4x+1 & 5-2x \\ 2-2x & x-2 \end{bmatrix}$. What is the largest entry of $\lim_{x \to \infty} A^{-1}$?

A. $\frac{1}{7}$ B. $\frac{2}{7}$ C. $\frac{4}{7}$ D. $\frac{8}{7}$ E. NOTA

12. Let $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$ be vectors. Let $\mathbf{v}_2$ rotate counterclockwise at a rate of $\frac{\pi}{32}$ radians per second and let $\mathbf{v}_1$ be fixed. At what rate is $\|\mathbf{v}_1 - \mathbf{v}_2\|$ changing after 8 seconds?

A. $\frac{\pi}{64}$ B. $\frac{\pi\sqrt{2}}{64}$ C. $\frac{\pi\sqrt{3}}{64}$ D. $\frac{\pi\sqrt{2}}{32}$ E. NOTA

13. Let A be a 3×2 matrix and let B be a 2×3 matrix. How many of the following products are defined?

    I. $A\cdot B$
    II. $B\cdot A$
    III. $B^{\mathrm{T}}\cdot B$
    IV. $A^{\mathrm{T}}\cdot A\cdot B\cdot B^{\mathrm{T}}$

    A. 1 B. 2 C. 3 D. 4 E. NOTA

14. Let R be the first quadrant region of the x-y plane bounded by the graphs $f(x)=2-x^2$, $g(x)=x^3$, and the y-axis. R is mapped to the region R' in the u-v plane by the transformation $u=5x-2y$ and $v=x+2y$. What is the area of R'?

    A. 7 B. 11 C. 17 D. 25 E. NOTA

15. Let $\mathbf{v}=(x,y,z)$ be a solution to the equations $x+z=6$ and $x+y+2z=12$. What is the smallest possible magnitude of $\mathbf{v}$?

    A. $3\sqrt{2}$ B. $2\sqrt{6}$ C. $3\sqrt{3}$ D. 6 E. NOTA

16. Let C be a curve given parametrically by $x=\sqrt{t^2+1}$ and $y=t^4+2t^2+4$ where $0\le t\le 1$. What is the area of the region beneath C and above the x-axis?

    A. $\frac{4\sqrt{2}}{5}$ B. $\frac{16}{5}$ C. $\frac{14\sqrt{2}}{5}$ D. 4 E. NOTA

17. Which of the following vectors is orthogonal to both $(1,0,1)$ and $(-1,2,1)$?

    A. $(0,1,0)$ B. $(2,1,0)$ C. $(1,-2,3)$ D. $(1,1,-1)$ E. NOTA

18. Let $A_n=\begin{bmatrix}\sqrt{n} & \sqrt{n+1}\\ \sqrt{n+1} & \sqrt{n+2}\end{bmatrix}$. What is $\lim\limits_{n\to\infty}|A_n|$?

    A. 0 B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. 1 E. NOTA

19. Let the vertices of a triangle in the *x*-*y* plane be the points $(0,0)$, $\left(4t-t^2,t\right)$, and $\left(\frac{t}{2},2\right)$, where $t\in[0,3]$. What is the maximum area of this triangle?

A. 0.75 B. 2 C. 3.2 D. 4 E. NOTA

20. Let *A* be a square matrix such that $A^{-1}=A^T$. Which of the following are true?

I. the columns of *A* are unit vectors
II. the columns of *A* are orthogonal
III. $\det A=1$
IV. $A=A^T$

A. I only B. I and II only
C. I, II, and III only D. I, II, III, and IV E. NOTA

21. Evaluate: $\begin{vmatrix} 1 & 0 & 5 \\ 0 & 4 & 1 \\ -2 & 3 & 2 \end{vmatrix}$.

A. 3 B. 31 C. 45 D. 98 E. NOTA

22. The line tangent to $f(x)=e^{\left(2-\sqrt{3}\right)x}-1$ when $x=0$ is rotated $30°$ counterclockwise about the origin. What is the slope of the rotated line?

A. $\frac{1}{2}$ B. 1 C. 2 D. 4 E. NOTA

23. If $2x+3y+z=-1$, $x+2y+3z=13$, and $5x-y-z=2$, then what is $x+y+z$?

A. -1 B. 1 C. 4 D. 7 E. NOTA

24. The position of a particle is given by $\mathbf{r}(t)=\left(3t^2+2t\right)\mathbf{i}+t\cos(t)\,\mathbf{j}+e^t\,\mathbf{k}$. What is the speed of the particle when $t=0$?

A. 1 B. $\sqrt{2}$ C. $\sqrt{3}$ D. $\sqrt{6}$ E. NOTA

25. Let $A=\begin{bmatrix} 1 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. Evaluate: $\lim_{n\to\infty}\sum_{i=1}^{n}\det\left(A^i\right)$.

A. $\frac{1}{6}$ B. $\frac{1}{5}$ C. $\frac{5}{6}$ D. 1 E. NOTA

26. What is the rank of $M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$?

A. 0 B. 1 C. 2 D. 3 E. NOTA

27. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Evaluate $\cos(A)$.

A. $\begin{bmatrix} \cos(1) & -\sin(1) \\ 0 & \cos(1) \end{bmatrix}$ B. $\begin{bmatrix} \cos(1) & \sin(1) \\ 0 & \cos(1) \end{bmatrix}$

C. $\begin{bmatrix} \cos(1) & -\cos(1) \\ 0 & \cos(1) \end{bmatrix}$ D. $\begin{bmatrix} \cos(1) & \cos(1) \\ 0 & \cos(1) \end{bmatrix}$ E. NOTA

28. Evaluate: $\begin{vmatrix} 2-3a & 3a+1 \\ a-2 & 1-a \end{vmatrix}$.

A. 0 B. 4 C. 12 D. 16 E. NOTA

29. Let $\mathbf{a} \oplus \mathbf{b} = \|\mathbf{a}+\mathbf{b}\| - \mathbf{a}\cdot\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are vectors. Calculate $(-1,3) \oplus (5,0)$.

A. 0 B. 2 C. 5 D. 10 E. NOTA

30. What is the distance between the point $(1,2,6)$ and the plane $2x - y + 2z + 3 = 0$?

A. $\sqrt{6}$ B. 3 C. 5 D. $3\sqrt{3}$ E. NOTA