2007 Mu Alpha Theta National Convention
Mu Number Theory Answers and Solutions

1) $2907 = 3^2 \cdot 17 \cdot 19$, $1596 = 2^2 \cdot 3 \cdot 7 \cdot 19$. gcd is $3 \cdot 19 = 57$ **C**
2) There are only three triplets (*A*, *B*, *C*) that work: (1, 9, 0), (5, 9, 2) and (6, 9, 3). **A**
3) 9 is always divisible by 999….9999 which is always one less than 10^n **B**
4) **D**
5) I, II, and IV are NOT prime. **D**
6) $10^3 < 32^2 < 33^2 < 34^2 < 35^2 < 36^2 < 11^3$ $32+33+34+35+36=170$ **B**
7) $2007^{2007}(\text{mod}100) \equiv 7^{2007}(\text{mod}100) \equiv 7^3(\text{mod}100) \equiv 43(\text{mod}100)$ **C**
8) For $(n-1)!$ to be divisible by *n*, it is equivalent to $n!$ divisible by n^2. This means that *n* cannot be prime. The only composite number that this doesn't hold true for is *n* = 4. There are 10 prime integers between 1 and 30 inclusive, thus there are 11 for which it is not valid. **C**
9) The only possible positive integers that has $d(n)=3$ are the squares of prime numbers. The ones less than 1000 would be: $2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2$ **A**
10) $200 = 2^3 \cdot 5^2$ In order for *n*! to not be congruent to $0(\text{mod}\,200)$, then it cannot contain all of the factors of 200. $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ which is congruent to $0(\text{mod}\,200)$ and $9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$ which is not congruent. Thus the largest value is 9 **E**
11) $1^1 \equiv 1(\text{mod}10)$, $2^2 \equiv 4(\text{mod}10)$, $3^3 \equiv 7(\text{mod}10)$, $4^4 \equiv 6(\text{mod}10)$, $5^5 \equiv 5(\text{mod}10)$, $6^6 \equiv 6(\text{mod}10)$, $7^7 \equiv 7^3(\text{mod}10) \equiv 3(\text{mod}10)$, $8^8 \equiv 8^4(\text{mod}10) = 6(\text{mod}10)$, $9^9 \equiv 9^1(\text{mod}10)$
$[1+4+7+6+5+6+3+6+9](\text{mod}10) = 43(\text{mod}10) \equiv 3(\text{mod}10)$ **C**
12) The number of zeros in 2007! is found by dividing 2007 by powers of 5:
$\lfloor 2007/5 \rfloor = 401$, $\lfloor 2007/25 \rfloor = 80$, $\lfloor 2007/125 \rfloor = 16$, $\lfloor 2007/625 \rfloor = 3$; $401+80+16+3=500$.
When raising a number to a power, we multiply the power by the number of zeros to get the total number of zeros: $500 \cdot 2 = 1000$ **D**
13) A theorem states that if we're given an order *q* of an element and the group of integers under multiplication (mod *p*) where *p* is prime, then $q \mid (p-1)$. Since $3 \nmid 10$, then there cannot be any elements of order 3. **A**
14) Any number with more than 3 digits will reduce the number of digits. Any number with 1 digit will increase the number of digits. Any number with 2 digits will give a number of 2 or 3 digits. When a cycle (or limit) occurs, all of the numbers must have the following property: must be 2 or 3 digits long, sum of hundreds and tens digits must be equal to units digit, and the units digit must be either 2 or 3. (This is because all numbers will lead to numbers of this form, and these numbers are closed.) We only have to consider the following numbers now: 22, 33, 112, 202, 123, 213, and 303. Noting that 33→22→202→303→123, 213→123, 112→123, and 123→123 proves that every number has a limit of 123. **D**
15) The Fibonacci numbers less than 2007 are
1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597. The ones that satisfy ${F_n}^2+1$ being divisible by ten must have a 3 or a 7 in the one's digit: 3, 13, 233, 377, 987, 1597. **D**
16) We solve this similar to a base-26 problem. $2007(\text{mod}\,26) = 5$, $(2007-5)/26 = 77$
$77(\text{mod}\,26) = 25$, $(77-25)/26 = 2$. $2 \Rightarrow b$, $25 \Rightarrow y$, $5 \Rightarrow e$ **C**
17) The largest odd square less than 2007 is $43^2 = 1849$. $2(22)-1 = 43$

Mu Number Theory Solutions

$1^2+3^2+5^2+\ldots=\sum_{n=1}^{22}(2n-1)^2=\sum_{n=1}^{22}4n^2-4n+1=\frac{4(22)(23)(45)}{6}-\frac{4(22)(23)}{2}+22(1)=14190$ **A**

18) $723_8=7(64)+2(8)+3=467_{10}, 124_5=25+2(5)+4=39_{10}, 723_8+124_5=506_{10}$,
$506_{10}=622_9$ **B**

19) Since 102! = $102\cdot101\cdot100!$, the greatest common divisor is 100! **E**

20) Simplifying the expression yields $m!=\frac{20n!}{n!-20}$. For n larger than 6, the ratio $\frac{20n!}{n!-20}$ will not be an integer and n cannot be smaller than 4. For n = 4 we have $m!=120\Rightarrow m=5$. For n = 5 we have $m!=120\Rightarrow m=4$. Since we can switch m and n we have two solutions. **C**

21) In order to have an exact number of cents, our cost must be a multiple of $0.25. Let's denote this as 25n where n is the number of quarters we need. In order to have an exact number of dollars, when we multiply our cost without tax to 1.04, we should get an integer: 1.04(25n) = 26n, which must be divisible by 100; or 13n must be divisible by 50. This yields n = 50, so our pre-tax cost is 25(50) = 1250 cents = $12.50. **A**

22) I is TRUE since all primes greater than 3 are odd and $a\equiv-1(\text{mod } p)$ would correspond to an even integer. II is TRUE from p being odd and is trivially true for p = 2. III is TRUE by Wilson's Theorem. IV is TRUE from Fermat's Little Theorem. ALL are true. **A**

23) $7056=2^4\cdot3^2\cdot7^2$. Thus the smallest value for n is $2^2\cdot3\cdot7=84$ **C**

24) $4155=5\cdot6!+4\cdot5!+3\cdot4!+0\cdot3!+1\cdot2!+1\cdot1!$ 543011 **D**

25) The pattern is the following: 15, 28, 39, 48, 55, 60, 63, 64, 72, 72, 75, 76, 78, 78, 78 **B**

26) When first eliminating all the integers with a 1, there will be $9^3=729$ numbers remaining. If he was to eliminate all integers with a 2 from the remaining, there would be $8^3=512$ numbers remaining. However, he keeps all numbers with BOTH a 2 and a 3. The possibilities are: X23, X32, 2X3, 3X2, 23X, and 32X, where X is any digit from 0-9, excluding 1. This means there are 54 possibilities, but we have double counted the following numbers: 223, 232, 322, 332, 323, and 233. Subtracting these gives 48 numbers with BOTH a 2 and a 3 from the remaining. Adding these back in gives $512+48=560$. This means he crossed out $1000-560=440$ **A**

27) Trying the first few: $a_1=1$, $a_2=3$, $a_3=12$, $a_4=60$, … The terms are related to the factorial sequence by division of 2 and shifted to the left by 1. Thus $a_n=\frac{(n+1)!}{2}$ **C**

28) 2007 minutes is equivalent to 33 hours and 27 minutes. If we move forward by exactly 24 hours, the time remains the same. By moving 4 hours forward, we reach 0:07. Thus we must move 5 hours and 27 minutes forward from 0:07, which corresponds to 5:34 **B**

29) $28x\equiv2(\text{mod}\,54)$ is the same thing as saying: find an integer solution to $28x+54y=2$, a linear Diophantine equation. This is equivalent to $14x+27y=1$. Using the Euclidian algorithm, an initial solution is $x_0=2$ and $y_0=-1$. All possible solutions are in the form $x=2+27t$ and $y=-1-14t$. The integer values of x less than 100 are 2, 29, 56, and 83. **D**

30) $56=2^3\cdot7$; $72=2^3\cdot3^2$; The LCM is $2^3\cdot3^2\cdot7=504$ **C**