2007 Mu Alpha Theta National Convention
Mu Sequences and Series-SOLUTIONS

C 1.) $f(x) = \log e$. $1 + \log e + (\log e)^2 + ... = \frac{1}{1-\log e}$.

A 2.) $f(x) = \ln 10$. $\ln 10 + 0 + 0 + 0 + ... = \ln 10$.

D 3.) $1 + \frac{i}{2} - \frac{1}{4} - \frac{i}{8} + \frac{1}{16} + ... = (1 - \frac{1}{4} + \frac{1}{16} - ...) + i(\frac{1}{2} - \frac{1}{8} + ...)$. Since these are both geometric: $\frac{1}{1+\frac{1}{4}} + \frac{\frac{1}{2}}{1+\frac{1}{4}}i = \frac{4}{5} + \frac{2}{5}i$.

B 4.) $f(0) = 1$. $f'(x) = e^x \sin x + e^x \cos x, f'(0) = 1$. $f''(x) = 2e^x \cos x, f''(0) = 2$. $f'''(x) = 2e^x \cos x - 2e^x \sin x, f'''(0) = 2$. So $f(x) = 1 + x + x^2 + \frac{x^3}{3} + ....$ So the coefficient of the third order term is $\frac{1}{3}$.

E 5.) Since this is a geometric sum with ratio $(-\ln x)$, the sum, and thus the derivative, do not exist at $x = e + 1$.

B 6.) $f(x) = \cos x, g(x) = \sin x$. $(f(x) + g(x))^2 - g(2x) = ((\sin x)^2 + (\cos x)^2 + 2\sin x \cos x - \sin 2x) = 1$.

C 7.) A regular polygon with $n$ sides can be broken down into $n$ triangles, each sharing one side with the polygon and having a vertex at its center. Thus, the center angle of each triangle is $\frac{2\pi}{n}$. The height of each triangle, $x$, can be found, since $\frac{x}{1} = \cot \frac{\pi}{n}$. Thus, the area of each of the $n$ triangles is $\cot \frac{\pi}{n}$, so the area of the polygon is $n \cot \frac{\pi}{n}$.

C 8.) $\displaystyle\sum_{n=1}^{k} \frac{n(n+1)}{2} = \frac{k(k+1)(2k+1)}{12} + \frac{n(n+1)}{4}$. $\frac{100 \cdot 101 \cdot 201}{12} + \frac{100 \cdot 101}{4} = 25 \cdot 101 \cdot (67 + 1) = 171700$.

A 9.) It is possible for $f(x)$ to never converge. If the convergence intervals for $g(x)$ and $h(x)$ do not coincide, then it is possible that $\frac{g(x)}{k(x)}$ diverges on the interval where $h(x)$ converges and $\frac{h(x)}{k(x)}$ diverges on the interval where $g(x)$ converges.

E 10.) It is also possible for $f(x)$ to always converge. If the convergence interval of $k(x)$ is contained entirely within the convergence intervals of both $f(x)$ and $g(x)$, then it is possible that $\frac{g(x)h(x)}{k(x)}$ converges at all other values of $x$ as well.

C 11.) $\frac{g(x)}{k(x)}$ will necessarily converge on the interval where $g(x)$ converges, but it can diverge at all other points.

B 12.) This sum will have 3000 terms, 300 each of the numbers 1-10. Thus, $55 \cdot 300 = 16500$.

A 13.) $x^2 = f(g(x \cdot x^2))$. Since $f(g(y)) = y$, $x^2 = x^3$. The solutions are $x = 0$ and $x = 1$, the product is 0.

D 14.) $\dfrac{(\frac{n(n+1)}{2})^2}{\frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6}} = \dfrac{3}{2n+1}$.

E 15.) This series is geometric with ratio $\frac{2x+1}{3}$. $-3 < 2x + 1 < 3$. $x > -2$ and $x < 1$. The series doesn't converge at either endpoint, so $(-2, 1)$.

A 16.) Let $g$ equal Gaku's area. $200 = 100 + 10\sqrt{g} + g$. If $u = \sqrt{g}$, $0 = -100 + 10u + u^2$. Since $u > 0$, $u = 5\sqrt{5} - 5$. Thus, the difference between Aneesh's and Chen's areas is $150 - 50\sqrt{5}$.

B 17.) $f(x) = 6 + 4x + \frac{1}{1+2x}$. This series will converge if $-\frac{1}{2} < x < \frac{1}{2}$. Thus, $f'(x) = 4 - \frac{2}{(1+2x)^2}$. $9 \cdot f'(\frac{1}{4}) = 28$.

A 18.) If either $a < 0$ or $b < 0$, then there are points where the series will not exist, thus A is a necessary (but not sufficient) condition. B, C, and D are not necessary because we can allow one of $c$ or $d$ to be zero while keeping the other positive. $c \geq 0$ and $d \geq 0$ would also be necessary.

A 19.) $f(1) = 0$. $f'(x) = \frac{e^x}{x} + e^x \ln x - e$. $f'(1) = 0$. $f''(x) = \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x$. $f''(1) = e$. $f'''(x) = \frac{x^2e^x - 2xe^x + 2e^x}{x^3} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x$. $f'''(1) = 2e$. Since this is the term for the third derivative, the coefficient is $\frac{2e}{3!} = \frac{e}{3}$.

C 20.) Writing out the first few partial sums, approximating: $1 - \frac{1}{8} = 1 - .125 = .875$. $1 - \frac{1}{8} + \frac{1}{27} = 1 - .125 + .037 = .912$. Since the series is alternating and at this point is fluctuating by less than 0.05, the sum will round to 0.9.

B 21.) By partial fractions, $k\left(\sum_{n=2}^{k}\left(\frac{2}{n^2-1}\right) - \frac{3}{2}\right) = k\left(1 + \frac{1}{2} - \frac{3}{2} - \frac{1}{k} - \frac{1}{k+1}\right) = -\frac{k(2k+1)}{k(k+1)}$. Thus, $\lim_{k\to\infty} a_k = -2$.

C 22.) As $a_n$ converges to its limit, $a_n = \sqrt{12 - a_n}$, $a_n^2 + a_n - 12 = 0$. $(a_n - 3)(a_n + 4) = 0$. Since $a_n \geq 0$, $a_n$ converges to 3.

D 23.) Since the Taylor series for $g(x)$ converges for all real $x$, $f'(\frac{\pi}{4}) = g'(\frac{\pi}{4}) = \sqrt{2}$. Thus, $2 \cdot f'(\frac{\pi}{4}) = 2\sqrt{2}$.

E 24.) Since the Taylor series would be about $x = \frac{\pi}{2}$, the terms must have an $(x - \frac{\pi}{2})^n$ factor, not an $x^n$ factor.

C 25.) $a_1 = 0$. Making the substitution $u = \tan x$, $a_2 = 1$. Since every second term in the sum will be a one, and both the first and last terms are ones, the sum equals $\frac{k+1}{2}$.

A 26.) $a_n = \left(\frac{n^n(\sin n)^2}{(1+n)^n}\right)^n = \left(\frac{(\sin n)^2}{(1+\frac{1}{n})^n}\right)^n = \left(\frac{(\sin n)^2}{e}\right)^n = k^n$, where $0 \leq k < 1$. Thus, $a_n$ approaches 0.

B 27.) $f(x) = \frac{1}{1-x}$. $\int_0^{\frac{1}{2}} \frac{1}{1-x}dx = \ln 2$. This is closest to $\frac{1}{2}$.

E 28.) $\sum_{n=1}^{\infty} \frac{2^n(-1)^n}{n!} = (\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} - 1) = e^{-2} - 1$.

D 29.) $g(x) = 1 + 2 + 4 + ... + 2^x = (2^{x+1} - 1)$. Thus, $g'(6) = 2^7 \ln 2 = 128 \ln 2$.

B 30.) Converting the numbers to base ten, the sequence is $8, 15, 22, 29, 36, ....$ So, the next number in base ten is 43, which is 47 in base nine.