2007 Mu Alpha Theta National Convention
Mu Sequences and Series

Directions: For all questions, NOTA means none of the above answers is correct. Let $f(x)^n$ denote the n^{th} power of $f(x)$ and let $f^{(n)}(x)$ denote the n^{th} derivative of $f(x)$. Also, let $f^{(0)}(x) = f(x)$. Have fun!

1.) $f(x) = \frac{\log x}{\ln x}$. Find $\sum_{n=0}^{\infty} f(x)^n$.
a.) $\ln 10$
b.) $\log e$
c.) $\frac{1}{1-\log e}$
d.) $\frac{\ln 10}{1-\ln 10}$
e.) NOTA

2.) $f(x) = \frac{\ln x}{\log x}$. Find $\sum_{n=0}^{\infty} f^{(n)}(x)$.
a.) $\ln 10$
b.) $\log e$
c.) $\frac{1}{1-\log e}$
d.) $\frac{\ln 10}{1-\ln 10}$
e.) NOTA

3.) Find the sum of: $1 + \frac{i}{2} - \frac{1}{4} - \frac{i}{8} + \frac{1}{16} + ...$
a.) $1 - i$
b.) $1 + i$
c.) $\frac{4}{5} - \frac{2}{5}i$
d.) $\frac{4}{5} + \frac{2}{5}i$
e.) NOTA

4.) Find the coefficient of the third order term of the Maclaurin series expansion of $f(x) = e^x \sin x + 1$.
a.) 0
b.) $\frac{1}{3}$
c.) 1
d.) 2
e.) NOTA

5.) $f(x) = \ln x - (\ln x)^2 + (\ln x)^3 - (\ln x)^4 + ...$ Find $f'(e+1)$.
a.) 0
b.) $\frac{\ln(e+1)}{1-\ln(e+1)}$
c.) $\frac{1}{(e+1)(1-\ln(e+1))^2}$
d.) $\frac{1}{(e+1)(1-\ln(e+1))}$
e.) NOTA

6.) If $f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$ and $g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$, find $(f(x) + g(x))^2 - g(2x)$.
a.) 0
b.) 1
c.) $-f(2x)$
d.) $g(x)$
e.) NOTA

7.) Gabor has always wondered whether there is a formula for the area of a regular polygon with n sides of length 2. Which of these sequences, a_n, works as such a formula for $n > 2$? He thinks he's got it, but feel free to disagree. Dispute forms will be available after the test.
a.) $a_n = n\cot\left(\frac{(n-2)\pi}{n}\right)$
b.) $a_n = n\cot\left(\frac{2(n-2)\pi}{n}\right)$
c.) $a_n = n\cot\left(\frac{\pi}{n}\right)$
d.) $a_n = n\cot\left(\frac{2\pi}{n}\right)$
e.) NOTA

8.) Find the sum of the first 100 triangular numbers, where the n^{th} triangular number is given by the sum of the first n positive integers.
a.) 5050
b.) 169175
c.) 171700
d.) 242400
e.) NOTA

For questions 9-11, consider $f(x)$, $g(x)$, $h(x)$, and $k(x)$ to be series where $f(x) = \frac{g(x)h(x)}{k(x)}$. You may assume that $k(x) \neq 0$ for all real x.

9.) Given that the radius of convergence of $g(x)$ is 3, the radius of convergence of $h(x)$ is 4, the radius of convergence of $k(x)$ is 2, what is the smallest possible radius of convergence of $f(x)$?
a.) 0
b.) 1
c.) 3
d.) 7
e.) NOTA

10.) Given the same radii of convergence, what is the largest possible radius of convergence of $f(x)$?
a.) 0
b.) 1
c.) 7
d.) 9
e.) NOTA

11.) Given the same radii of convergence, other than $h(x)$, which is now a constant, what is the smallest possible radius of convergence of $f(x)$?
a.) 0
b.) 1
c.) 3
d.) 7
e.) NOTA

12.) Evaluate $\sum_{a=1}^{10}\left(\sum_{b=1}^{10}\left(\sum_{c=1}^{10}(a+b+c)\right)\right)$.
a.) 13500
b.) 16500
c.) 91125
d.) 166375
e.) NOTA

13.) Given that $f(x)$ is the inverse of $g(x)$, what is the product of all of the solutions to the following equation: $x^2 = f(g(xf(g(xf(g(xf(....)))))))$?
a.) 0
b.) -1
c.) 1
d.) There are no solutions.
e.) NOTA

14.) Which of the following is equal to $\dfrac{1^3+2^3+3^3+...+n^3}{(1+2+3+...+n)(1^2+2^2+3^2+...+n)}$ for all positive integers n?
a.) $\frac{3}{2}$
b.) $\frac{1}{n+1}$
c.) $\frac{7}{5}-\frac{2}{5}n$
d.) $\frac{3}{2n+1}$
e.) NOTA

15.) $f(x) = \sum_{n=0}^{\infty}\dfrac{(2x+1)^n(-1)^{n+1}}{3^{n-1}}$. On which of the following intervals does $f(x)$ converge absolutely?
a.) $(-1,2)$
b.) $[-1,2)$
c.) $(-2,1]$
d.) $[-2,1]$
e.) NOTA

16.) Certain mathletes, namely Aneesh, Gaku, and Chen, must share a room to take a test. Aneesh is kind of buff, so he needs exactly half of the room. Gaku is small and thus needs the least room of all three. Chen's only request is to have more room than Gaku, so he is assigned the positive geometric mean of Gaku's and Aneesh's spaces. If the room is exactly 200 units square, and Aneesh is willing to scale down to an area equal to Chen's, how much of the room could be given to Karlanna, who would like to enter this Mu battle of wits?
a.) $150-50\sqrt{5}$
b.) $50+50\sqrt{5}$
c.) $150+50\sqrt{5}$
d.) $5\sqrt{5}-5$
e.) NOTA

17.) $f(x) = 7+2x+4x^2-8x^3+16x^4-32x^5+64x^6-128x^7+256x^8-...$ Find $9\cdot f'(\frac{1}{4})$.
a.) 8
b.) 26
c.) 44
d.) 107
e.) NOTA

18.) If $f(x) = (\sin x)^a(\cos x)^b + (\sin x)^{a+c}(\cos x)^{b+d} + (\sin x)^{a+2c}(\cos x)^{b+2d} + ...$, which of the following is a necessary condition for $f(x)$ to converge and exist for all real x? Note that a, b, c, and d are integers.
a.) $a \geq 0, b \geq 0$
b.) $c > 0, d > 0$
c.) $c \geq 0, d > 0$
d.) $a > 0, b > 0, c > 0, d > 0$
e.) NOTA

19.) Find the coefficient of the second term of the Taylor series expansion of $f(x) = e^x \ln x - ex + e$ about $x = 1$.
a.) $\frac{e}{3}$
b.) $\frac{e}{2}$
c.) $\frac{2e}{3}$
d.) e
e.) NOTA

20.) Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ to the nearest tenth.
a.) -1.0
b.) -0.8
c.) 0.9
d.) 1.0
e.) NOTA

21.) Define $a_k = k\left(\displaystyle\sum_{n=2}^{k}\left(\frac{2}{n^2-1}\right) - \frac{3}{2}\right)$ for all $k \geq 3$. Find $\displaystyle\lim_{k\to\infty} a_k$.
a.) $-\infty$
b.) -2
c.) $\frac{1}{2}$
d.) 1
e.) NOTA

22.) Given that $a_n = \sqrt{12 - a_{n-1}}$ and $a_1 = 12$, find $\displaystyle\lim_{n\to\infty} a_n$.
a.) -4
b.) 0
c.) 3
d.) 12
e.) NOTA

23.) Let $f(x)$ be the Taylor Series expansion for the function $g(x) = \sin x - \cos x$. Find $2 \cdot f'(\frac{\pi}{4})$.
a.) $-2\sqrt{2}$
b.) 0
c.) $\sqrt{2}$
d.) $2\sqrt{2}$
e.) NOTA

24.) Find the third term of the Taylor series expansion of $f(x) = x \sin x$ about $x = \frac{\pi}{2}$.
a.) $-\frac{\pi x^2}{4}$
b.) $-\frac{\pi^2 x^2}{8}$
c.) $-\frac{x^3}{2}$
d.) $-\frac{\pi x^3}{4}$
e.) NOTA

25.) Consider the sequence defined by $a_1 = \lim_{x \to 0^+} \tan x$, $a_2 = \lim_{x \to 0^+} \tan x^{\tan x}$, $a_3 = \lim_{x \to 0^+} \tan x^{\tan x^{\tan x}}$, so on. You may assume that all even terms are equal and all odd terms are equal. Thus, $a_1 = a_3$ and so on. Find $\sum_{n=1}^{k} a_n$ for odd k.
a.) 0
b.) $\frac{k-1}{2}$
c.) $\frac{k+1}{2}$
d.) ∞
e.) NOTA

26.) Find $\lim_{n \to \infty} a_n$ if $a_n = \left(\frac{n^n (\sin n)^2}{(1+n)^n} \right)^n$.
a.) 0
b.) 1
c.) ∞
d.) Does not exist.
e.) NOTA

27.) Which of the following is the closest to the area under the curve $f(x) = 1 + x + x^2 + x^3 + x^4 + ...$ on the interval $[0, \frac{1}{2}]$?
a.) 0
b.) $\frac{1}{2}$
c.) 1
d.) ∞
e.) NOTA

28.) Evaluate $\sum_{n=1}^{\infty} \frac{2^n (-1)^n}{n!}$.
a.) $\ln 2$
b.) $e - 1$
c.) e^{-2}
d.) $\frac{e^2}{2} - 1$
e.) NOTA

29.) Define $f(x,y)$ as the value of the y^{th} term of the x^{th} row of Pascal's triangle. Also, define $g(x) = \sum_{a=0}^{x}\sum_{b=0}^{a} f(a,b)$. If we assume $g(x)$ to be differentiable for all $x \geq 0$, what is $g'(6)$? Note that the top row of Pascal's triangle is defined as the zeroth row, and the first term in each row is the zeroth term.
a.) 127
b.) $63 \ln 2$
c.) $64 \ln 2$
d.) $128 \ln 2$
e.) NOTA

30.) All numbers in this problem (after this first one, and including the answers) are in base nine. What is the next number in the following pattern: $8, 16, 24, 32, 40, ...$?
a.) 44
b.) 47
c.) 48
d.) 53
e.) NOTA