Theta Applications Solutions
Mu Alpha Theta Nationals, 2007

1.   Answer: B   17
The rectangular piece of paper has two dimensions, width and height. When the cylinder is formed, its height is one of these two dimensions, and the circumference of its base is the other. This is shown in the diagram below (not by any means drawn to scale). Therefore, the dimensions of the original piece of paper were 15 cm and 8 cm, and its diagonal, found via the Pythagorean Theorem, is 17 centimeters.
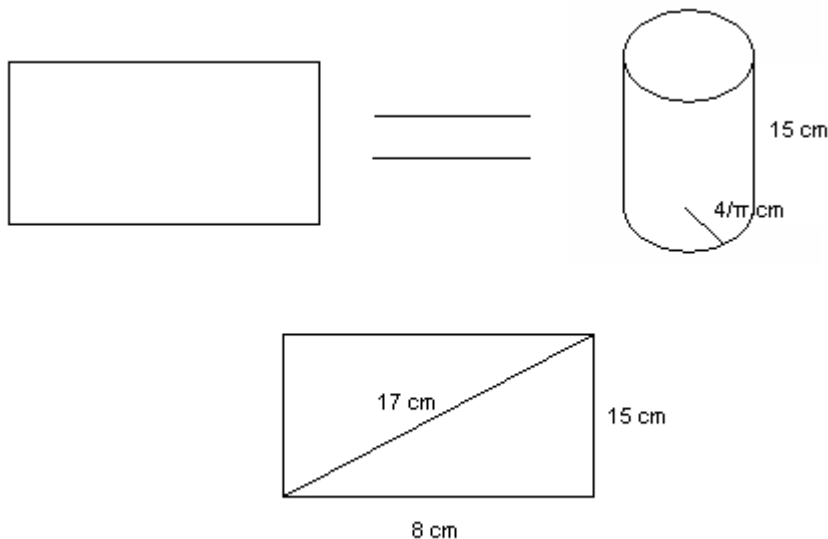
2.   Answer: C $3.75
Three hot dogs and one hamburger cost $2.25, and one hot dog and three hamburgers cost $2.75. Solving the system yields prices of $0.50 for one hot dog and $0.75 for one hamburger. Therefore, the combined cost of one of each item is $1.25, and paying $5 will yield $3.75 in change.

3.   Answer: A   5/27
The value on the die will either be 1, 2, 3, 4, 5, or 6. The value of the card will either be 2, 3, 4, 5, 6, 7, 8, 9, or 10. Thus, the die roll is higher than the card draw only for the paired results (3,2), (4,2), (4,3), (5, 4), (5,3), (5,2), (6,5), (6,4), (6,3), and (6,2). The probability of any one of these results occurring is $\frac{1}{6} \times \frac{4}{36}$, or $\frac{1}{54}$. The probability that one of the ten will occur is $10 \times \frac{1}{54} = \frac{5}{27}$.

4.   Answer: D $y = \log(x + 17) - 2$
Where defined, the inverse of the function $y = 10^{(x+2)} - 17$ is $x = 10^{(y+2)} - 17$. Simplified, this yields $x + 17 = 10^{(y+2)}$, or $\log(x + 17) = y + 2$. Thus, $y = \log(x + 17) - 2$.

5. Answer: C $4\sqrt{6}$

If the window has width $\sqrt{6}$ and height Y, then the doorway has width $\dfrac{Y}{3}$ and height $2\sqrt{6}$. As the rectangles are similar, the ratio of $\sqrt{6}$:Y equals the ratio of $\dfrac{Y}{3}$: $2\sqrt{6}$. Thus, Y equals 6. The doorway therefore has dimensions of 2 feet and $2\sqrt{6}$ feet, and its area is $4\sqrt{6}$ square feet.

6. Answer: B $\dfrac{79}{4}\pi$

Skippy can graze within a circle of radius 5 yards from the corner of the barn where he is tied. As the barn obstructs a quarter of this circle, the area is cut to $\dfrac{3}{4}$ of that, or $\dfrac{75}{4}\pi$ square yards. However, the rope is longer than the barn on one side by two yards, and thus Skippy can also graze in an additional quarter circle around the side of the barn. This quarter circle has radius 2 yards, and thus has an area of π. Therefore, Skippy's total grazing area is $\dfrac{79}{4}\pi$ square yards.
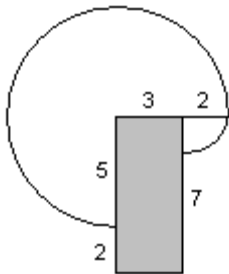
7. Answer: E 3
At 5:30 pm, people have been leaving the park for 5.5 hours, for a total of 121 people leaving. People have been entering the park for 4.5 hours at this point, for a total of 90 people entering the park. Thus, at 5:30 there are 31 fewer people in the park than there were at 12:00 pm. As there were 34 in the park at 12:00, at 5:30 there are only 3.

8. Answer: C 55
At most, 87.5% of the members will show up to vote. Assuming all of them favor the same candidate, they provide 658 votes for him or her. Another 44 votes are required for the candidate's election, and since each mail-in ballot counts as 80% of a vote, 55 mail-in ballots are needed to win the election.

9.   Answer: B   2

The volume of material remains constant throughout the molding process. Therefore, the volume of the sphere is equal to the combined volume of the two cylinders. The sphere, with radius 3, has a volume $\frac{4}{3}\pi r^3 = 36\pi$. Thus, $36\pi = 2(\pi r^2 h)$, and as the cylinders have a radius of 3, h = 2.

10.   Answer: E   "If it is Alton's birthday, then Cole plays football."

The converse of a statement is merely the reversal of its conditional and resultant clauses, without negation. Therefore the converse of "If A, then B" is "If B, then A," and therefore the converse of the given statement is "If it is Alton's birthday, then Cole plays football."

11.   Answer: C. 600π

The circumference of the wheel is 12π inches, or π feet. As the wheel makes ten complete revolutions each minute, the unicycle travels ten times the circumference every minute, or 10π feet. Therefore, in one hour, the unicycle travels 600π feet.

12.   Answer: A.   Row 14

Each row in the Infinite Theatre has a composite number of seats, starting with 4 seats in Row 1 and with each successive row having the next highest composite number of seats. Thus, Row 11 has 20 seats, Row 12 has 21 seats, etc. As Tyler is losing two friends per row, he has 23 friends with him when he reaches Row 14, for a total of 24 people in his party (counting Tyler). This is the right row, for Row 14 holds exactly 24 seats, 24 being the 14th smallest positive composite number.

13.   Answer: D   Sixteen minutes, ten seconds

The sum of an infinite geometric series can be found by dividing the first term by (1 – r), where r is the common ratio. Thus, as the first musical number in the opera took 194 seconds, and the common ratio is $\frac{4}{5}$, the total length of the show is $194 \div \frac{1}{5} = 970$ seconds, or sixteen minutes, ten seconds.
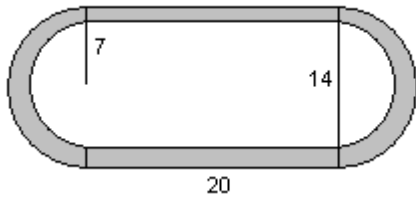

14.   Answer: B   1260

$$\frac{(\text{Total number of letters})!}{(\text{Number of Ns})! \times (\text{Number of Os})! \times (\text{Number of Ys})! \times (\text{Number of Es})! \times (\text{Number of Ss})!}$$

$$\frac{7!}{(2!) \times (2!) \times (1!) \times (1!) \times (1!)} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (2 \times 1) \times (1) \times (1) \times (1)} = \frac{5040}{4} = 1260$$

15.   Answer: B $80 + 24\pi$

Like the swimming pool, the outline of the flower garden is made up of two semicircles and a connecting rectangle. The area of the garden itself is the area enclosed by these minus the area of the pool itself. The area enclosed equals the area of the two semicircles and the area of the rectangle, or $\frac{49}{2}\pi + \frac{49}{2}\pi + (14 \times 20) = 280 + 49\pi$ square feet. The area of the pool, made up of the two smaller semicircles and the smaller rectangle, is $\frac{25}{2}\pi + \frac{25}{2}\pi + (10 \times 20) = 200 + 25\pi$ square feet. Thus, the total area of the flower garden is $80 + 24\pi$ square feet.

16.   Answer: B $\sqrt{5}$

Given four relatively prime numbers, no more than one of the numbers may be even- for if two or more were even, they would have a common factor of 2. As inches South are essentially negative inches North, and inches West are essentially negative inches East, N should be as close to S as possible, and E as close to W. As one number may be even, its counterpart should be either the odd number directly higher than it or the odd number directly lower than it. For example, N and S could be 9 and 10, respectively. For the other two numbers, both must be odd, and thus the difference between them must be a minimum of two inches. Numbers such as 11 and 13, then, can apply for E and W, for in addition to being close, they are relatively prime with both each other and 9 and 10, our values for N and S. As total distance traveled equals $\sqrt{(E-W)^2 + (N-S)^2}$ inches, we can see that the smallest possible distance between Sammy's starting and ending positions is $\sqrt{5}$ inches.

17.   Answer: C $\frac{21}{52}$

The given probabilities are:

Chance of coin landing on heads $= \frac{1}{2}$

Chance of drawing a nonface card $= \frac{10}{13}$

Chance of rolling a prime number $= \frac{1}{2}$

Chance of rain $= \frac{1}{2}$

So, the probability that a card will be drawn is $\frac{1}{2}$. The probability that the die will be rolled is $\frac{1}{2} \times \frac{10}{13} = \frac{5}{13}$. The chance that a prime number will show, and thus that Jon will wear a raincoat today, is $\frac{5}{13} \times \frac{1}{2} = \frac{5}{26}$. Therefore, the probability that Jon will not wear a raincoat is $\frac{21}{26}$. There is a fifty percent chance of rain today, so the probability that both it will rain and Jon will not be wearing a raincoat (two mutually independent events) is $\frac{1}{2} \times \frac{21}{26} = \frac{21}{52}$.

18.   Answer: C   Day 99

The sum of an arithmetic sequence equals $\frac{n}{2} \times [2a + (n-1) \times d]$, where n is the number of terms (days, in this case), a is the value of the first term (the three dollars found on Day 1), and d is the amount each term is greater than the previous (two dollars). As Todd already has $1 in his possession, he needs another $9,999 to pay off his debt. Therefore, $9,999 = \frac{n}{2} \times [(2 \times 3) + (n-1) \times 2]$. Thus, $n^2 + 2n - 9999 = 0$. Solving for n finds that n could equal 99 or -101. As the day number must be positive, Todd will have enough money on the 99th day.

19.   Answer: B   Alpha

The first brother is either Mu (always honest), Alpha (never honest), or Theta (sometimes honest). He tells you he is Alpha. If he were Mu, he would never say such a thing, for it is untrue. Similarly, if he were Alpha, he would never tell you so, for it is the truth. Therefore, the first brother must be Theta. The second brother tells you that the first is not Alpha, which is a true statement. As only Mu and Theta can tell the truth, and the first brother is Theta, the second must be Mu. Therefore, the lying Alpha is the only one not present.

20.   Answer: B   1445

In his final step, the smallest number Mark could round to 2000 would be 1500. The smallest number that would round to 1500 in the step before that would be 1450. And finally, the smallest number that would round to 1450 in the first step is 1445.

21.   Answer: A 450_{11}

Brad receives a total of $\$133_{10} + \$133_9 + \$133_8 + \$133_7 + \$133_6 + \$133_5 + \$133_4$, which in base-ten is $\$133 + \$111 + \$91 + \$73 + \$57 + \$43 + \$31 = \$539$. $539_{10} = 450_{11}$.

22.   Answer: D   24

The tangent of the angle is $\frac{3}{4}$, therefore the ratio of the opposite side to the adjacent side to the hypotenuse is 3:4:5. As the semiperimeter of the triangle is 12, the sides are of lengths 6, 8, and 10, respectively. Therefore, the area of the triangle is $\frac{1}{2} \times 6 \times 8 = 24$.

23.   Answer: C   6

Chelsea's original mixture is 25% sugar, and she has 20 ounces of it. Therefore, there are 5 ounces of sugar in Chelsea's lemonade. She wants these 5 ounces to only be 20% of the total mixture, therefore her finished product will be 25 ounces total when mixed. Thus, Chelsea must add 5 ounces of pure hydrochloric acid to the mixture to reach this amount. As there was already 1 ounce of hydrochloric acid present before this new addition (5% of the original 20 ounces), Chelsea's finished version will have 6 ounces of hydrochloric acid.

24.   Answer: C   Five minutes

In three minutes, Arielle can eat eleven scoops of ice cream, and Lissa can eat ten. Therefore, together they can eat 21 scoops in three minutes, or 7 scoops a minute. As the UltraMegaHurriQuakeNado consists of 35 scoops, it would take the pair a mere five minutes to finish it.
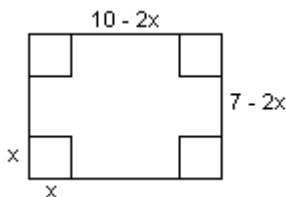
25.   Answer: D   9

Doug is cutting away x inches from each side, as shown in the diagram to the left. He will make his box by folding up the resulting flaps, and therefore the base of the box will be (10 – 2x)(7 – 2x), or $(4x^2 – 34x + 70)$ square inches in area. As the area of the base is to be 28 square inches, $4x^2 – 34x + 70 = 28$, or $4x^2 – 34x + 42 = 0$. Solving for x finds possible values of 1.5 and 7, but of course Doug cannot cut 14 inches off sides that measure 7 and 10 inches, respectively. Therefore, x = 1.5, and thus Doug cuts away four squares of area $(1.5) \times (1.5)$ square inches. $(1.5) \times (1.5) \times 4 = 9$ square inches.

26.   Answer: D   300

Each cone has radius 3 cm and height 5 cm, and thus volume $\frac{1}{3} \times 3^2 \times 5 \times$ π, or 15π cm³.

The pumpkin, being a sphere, has a volume of $\frac{4}{3} \times 15^3 \times$ π, or 4500π cm³. Therefore, the pumpkin will hold $4500\pi \div 15\pi$, or 300 ice cones.

27.   Answer: A   112
30x + 40y = 8050
x + y = 231
x = 119, y = 112

28.   Answer: A $50\pi\sqrt{3}$
Lassie can travel to any point at which the combined distance to the two pegs does not exceed 20 feet. Thus, an ellipse, with the two pegs as foci. With the rope fully extended along the same axis as the foci, Lassie can go a maximum of 5 feet from one peg, 15 feet from the other (there are ten feet of slack rope once Lassie passes the second peg; she can therefore go five feet further). This is shown in Figure 1 below. Therefore, a, marked in the figure, equals 10. When the rope is fully extended along an axis perpendicular to the axis joining the pegs, Lassie can go a maximum of b feet, as shown in Figure 2. Using the Pythagorean Theorem, $b = 5\sqrt{3}$. The ellipse's major and minor axes are 2a and 2b respectively, and the area of an ellipse is abπ. Therefore, the area of the region in which Lassie can travel is $50\pi\sqrt{3}$ square feet.

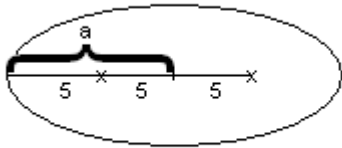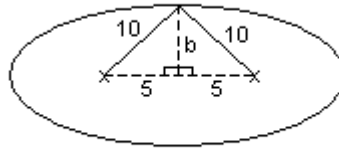Fig. 1   Fig. 2

29.   Answer: B $6
If $x = \sqrt{30 + \sqrt{30 + \sqrt{30 + \sqrt{30 + ...}}}}$, the number of dollars Horace will lose, then $x^2 = 30 + \sqrt{30 + \sqrt{30 + \sqrt{30 + \sqrt{30 + ...}}}}$, or 30 + x. Therefore, $x^2 - x - 30 = 0$. Solving for x yields values of -5 and 6, but $\sqrt{30 + \sqrt{30 + \sqrt{30 + \sqrt{30 + ...}}}}$ must be a positive number, as it is a square root. Therefore, assuming you answered correctly, Horace now owes you six dollars. If he doesn't believe you, show him this explanation.

30.   Answer: A   1.5
As time progresses, red balloons are being added to the sky and bottles of ginger ale are being taken down off the wall. There are 99 balloons 6 hours after there were only 15, so the number of red balloons is growing at a rate of 14 balloons every hour. There are also 71 bottles 6 hours after there were 99, so the number of bottles is decreasing at a rate of 14 bottles every three hours. So for a given time t hours ago, there were 99 – 14t red balloons floating in the summer sky, and $71 + 14\dfrac{t}{3}$ bottles of ginger ale on the wall.

Setting these equations equal to one another finds that the number of balloons and the number of bottles were each 78 exactly 1.5 hours ago.