2007 Mu Alpha Theta National Convention
Theta Sequences and Series Answers and Solutions

Answer Key

1)	A
2)	D
3)	B
4)	B
5)	C
6)	A
7)	D
8)	D
9)	A
10)	C
11)	C
12)	C
13)	E
14)	D
15)	C
16)	D
17)	B
18)	C
19)	D
20)	B
21)	D
22)	E
23)	B
24)	B
25)	D
26)	C
27)	D
28)	A
29)	A
30)	D

Theta Sequences and Series Solutions

1) $-14-(-5)=-9$ **A**

2) $\dfrac{-5/4}{-25/12}=\left(-\dfrac{5}{4}\right)\left(-\dfrac{12}{25}\right)=\dfrac{3}{5}$ **D**

3) $a_5=2$, $a_{13}=8$, $a_{13}-a_5=8d=8-2\Rightarrow d=3/4$, $2=a_1+3/4(5-1)\Rightarrow a_1=-1$,
$S_9=9/2(2(-1)+3/4(9-1))=18$ **B**

4) $a_3=2$, $a_7=8$, $a_7/a_3=r^4=8/2\Rightarrow r=\sqrt{2}$, $2=a_1\cdot(\sqrt{2})^2\Rightarrow a_1=1$,

$S_6=\left(\dfrac{1-(\sqrt{2})^6}{1-\sqrt{2}}\right)=\dfrac{-7}{1-\sqrt{2}}=7(1+\sqrt{2})=7\sqrt{2}+7 \qquad 7+2+7=16$ **B**

5) $X_1=4$, $X_2=2\sqrt{3}$ $\quad \dfrac{4}{2\sqrt{3}}=\dfrac{2\sqrt{3}}{3}$ **C**

6) 1: one 1, one digit; 101: two 1s, three digits; 101001: three 1s, six digits; etc. Let n be the number of 1s in the number, then there are $\dfrac{n(n+1)}{2}$ digits. $\dfrac{25(26)}{2}=325$ **A**

7) The distances traveled north are in a geometric sequence and the distances traveled southeast are also in a geometric sequence.
Total distance =
$(1+\sqrt{2})+(1/2+\sqrt{2}/2)+(1/4+\sqrt{2}/4)+\ldots=(1+1/2+1/4+\ldots)+(\sqrt{2}+\sqrt{2}/2+\sqrt{2}/4+\ldots)=$
$2+2\sqrt{2}$ **D**

8) $60=\dfrac{12}{2}(2(-6)+d(12-1))\Rightarrow d=2 \qquad S_{20}=\dfrac{20}{2}(2(-6)+2(20-1))=260$ **D**

9) Total men = $\dfrac{30}{2}[2(10)+8(30-1)]=3780$, Total women = $\dfrac{30}{2}[2(20)+16(30-1)]=7560$.
Difference = 3780 **A**

10) The first 4 should be known: $1^1=1$; $2^2=4$; $3^3=27$; $4^4=256$. The units digit of 5 to any power is 5 and the units digit of 6 to any power is 6. 7, 8, and 9 have patterns:
units digits for 7: $7^1\to 7$; $7^2\to 9$; $7^3\to 3$; $7^4\to 1$; $7^5\to 7$ $\qquad 7^7=7^4\cdot 7^3\to 3\cdot 1=3$
units digits for 8: $8^1\to 8$; $8^2\to 4$; $8^3\to 2$; $8^4\to 6$; $8^5\to 8$ $\qquad 8^8=8^4\cdot 8^4\to 6^2\to 6$
units digits for 9: $9^1\to 9$; $9^2\to 1$; $9^3\to 9$; $9^4\to 1$; $9^5\to 9$ $\qquad 9^9=9^4\cdot 9^5\to 9\cdot 1=9$
$1+4+7+6+5+6+3+6+9=47\to 7$ **C**

11) Area of largest semicircle = $\dfrac{\pi}{2}R^2$; Area of next largest = $\dfrac{\pi}{2}\left(\dfrac{R}{2}\right)^2=\dfrac{\pi}{8}R^2$; next largest = $\dfrac{\pi}{32}R^2$
Area of shaded region =
$\dfrac{\pi}{2}R^2-\dfrac{\pi}{8}R^2+\dfrac{\pi}{32}R^2-\dfrac{\pi}{128}R^2+\ldots=\dfrac{\pi}{2}R^2\left[1-\dfrac{1}{4}+\dfrac{1}{16}-\dfrac{1}{64}+\ldots\right]=\dfrac{\pi}{2}R^2\left[\dfrac{1}{1-(-1/4)}\right]=\dfrac{2\pi}{5}R^2$ **C**

12) $\dfrac{1}{3}\cdot\dfrac{2}{4}\cdot\dfrac{3}{5}\cdot\ldots\cdot\dfrac{97}{99}\cdot\dfrac{98}{100}\cdot\dfrac{99}{101}=1\cdot 2\cdot\dfrac{3}{3}\cdot\dfrac{4}{4}\cdot\ldots\cdot\dfrac{98}{98}\cdot\dfrac{99}{99}\cdot\dfrac{1}{100}\cdot\dfrac{1}{101}=\dfrac{2}{(100)(101)}=\dfrac{1}{5050}$ **C**

13) 8:00am–10:00pm=840 min. 1st cycle=18min. 2nd cycle=36min $18+36+72+144+288=558$,
$18+36+72+144+288+576=1134$ There are 5 complete cycles. **E**

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14) This is a geometric series with a common ratio of $\frac{i+1}{2}$. The infinite sum is $\frac{1}{1-(i+1)/2}=1+i$ **D**

15) Let $a_n=\left(\frac{1}{2}\right)^n$ and $b_n=\left(\frac{1}{3}\right)^n$. $\{a_n\}$ and $\{b_n\}$ are convergent geometric sequences.

$c_n=\left(\frac{1}{2}\right)^n+\left(\frac{1}{3}\right)^n$. I) False. $\{c_n\}$ does not necessarily have a constant common ratio. II) False. $\{c_n\}$ does not necessarily have a constant common difference. III) True. The sum of two convergent sequences is also convergent. **C**

16) For a quadratic sequence, the difference between terms increases linearly. Thus the difference is decreasing by 7 according to the triangle:

$$\begin{array}{ccccc} \underline{4} & & 7 & & 3 \\ & \underline{3} & & -4 & \\ & & \underline{-7} & & \end{array}$$

The quadratic expression for each term is: $4+3(n-1)-\frac{7}{2}(n-1)(n-2)=-\frac{7}{2}n^2+\frac{27}{2}n+6$ $\quad \frac{7}{2}+\frac{27}{2}+6=23$ **D**

17) $a_3=1-(-2)=3$, $a_4=3-1=2$, $a_5=2-3=-1$, $a_6=-1-2=-3$, $a_7=-3-(-1)=-2$ **B**

18) Let $x=\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}$ $\quad x^2=6+x\Rightarrow x^2-x-6=0\Rightarrow x=-2,3;$ x must be positive **C**

19) $1^2+3^2+5^2+\ldots=\sum_{n=1}^{10}(2n-1)^2=\sum_{n=1}^{10}4n^2-4n+1=\frac{4(10)(11)(21)}{6}-\frac{4(10)(11)}{2}+10(1)=1330$ **D**

20) This is a geometric series with a ratio of $1/3$ and the first term is $1/2$. $\frac{1/2}{1-1/3}=\frac{3}{4}$ **B**

21) $\sum_{n=1}^{2007}\log_3(n+1)=\log_3 2+\log_3 3+\ldots+\log_3 2007+\log_3 2008=\log_3(2\cdot 3\cdot\ldots\cdot 2008)=\log_3(2008!)$ **D**

$2\log_3(20!)\approx 77$ **C**

22) $\frac{2/27}{1/4}=\frac{8}{27}=r^3\Rightarrow r=\frac{2}{3}$ $\quad a=\frac{1}{4}\cdot\frac{2}{3}=\frac{1}{6}$, $b=\frac{1}{6}\cdot\frac{2}{3}=\frac{1}{9}$ $\quad \frac{1}{6}+\frac{1}{9}=\frac{5}{18}$ **E**

23) The ball falls 12 feet, rebounds 12/5 feet to a maximum height, then falls another 12/5 feet back down to the ground, and so forth. The total distance is: $12+2\left(\frac{12}{5}\right)+2\left(\frac{12}{5^2}\right)+\ldots$

$=12+\sum_{n=1}^{\infty}2\left(\frac{12}{5^n}\right)=12+24\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n=12+24\frac{1/5}{1-1/5}=18$ **B**

24) $a_1=\pi$, $d=e-\pi$ $\quad a_{10}=\pi+(e-\pi)(10-1)=9e-8\pi$ **B**

25) I) True $\sum_{n=0}^{\infty}a\cdot b^n=\frac{a}{1-b}$, II) True $\sum_{n=0}^{\infty}\frac{1}{a\cdot b^n}=\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{1}{b}\right)^n$ which diverges because $\frac{1}{b}>1$

III) True, $a\cdot b^n$ is always less than 1 for any positive integer *n*.

IV) True $\sum_{n=0}^{\infty}(a\cdot b^n)^2=a^2\sum_{n=0}^{\infty}(b^2)^n=\frac{a^2}{1-b^2}<\frac{a}{1-b^2}=\frac{a}{(1+b)(1-b)}<\frac{a}{1-b}=\sum_{n=0}^{\infty}a\cdot b^n$ **D**

26) The common difference is not constant (not arithmetic) and the common ratio is not constant (not geometric). **C**

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27) $\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{9}\right)+\left(\frac{1}{8}-\frac{1}{27}\right)+\ldots=\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)-\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\right)$

$=\frac{1/2}{1-1/2}-\frac{1/3}{1-1/3}=1-\frac{1}{2}=\frac{1}{2}$ **D**

28) The coefficient would be: $\binom{5}{2}(3x^4)^2(-4y)^3=-5760x^8y^3$ **A**

29) $\sum_{n=1}^{2007}\log\left(\frac{n}{n+1}\right)=\log\left(\frac{1}{2}\right)+\log\left(\frac{2}{3}\right)+\ldots+\log\left(\frac{2007}{2008}\right)=\log 1-\log 2+\log 2-\log 3+\log 3-\log 4+\ldots$

$+\log 2006-\log 2007+\log 2007-\log 2008=\log 1-\log 2008=-\log 2008$ **A**

30) The frog jumps on lily pads 1, 2, 3, 4, 5, 4, 5, 6, 7, 8, 9, 8, 9, 10, 11, 12, 13, 12, 13, … The lily pads that are hit more than once are: 4, 5, 8, 9, 12, 13, 16, 17, … We divide the sequence into two different arithmetic sequences: I) 4, 8, 12, 16, … II) 5, 9, 13, 17, … Let n be the n^{th} term of the second sequence. We want to find the largest value of n such that $4n+1<50 \Rightarrow n=12$. There are 12 terms in the second sequence, thus the frog jumps on 24 lily pads. **D**