For the following questions, select E, NOTA if none of the above answers is correct.

Good luck and have fun!

1. One day, Harish decided to pay a surprise visit to his friend's house. As he approached a door that he had never seen before, he noticed that the house number was 454. After Harish knocked on the door, his friend opened the door in surprise. He told Harish, "If you didn't know, 454 is my favorite number; I bet you haven't heard that before. Anyway, I'll let you come in if you solve this problem: let $f(x)$ be a polynomial function with nonnegative integral coefficients such that $f(101) = 454$. Compute the sum of all distinct possible values of $f(2019)$." Harish immediately said, "Oh dude, I memmed that the answer is __."

   Assuming Harish was let into the house, what answer did Harish give his friend?

   (A) 454 (B) 20996 (C) 21450 (D) 21904 (E) NOTA

2. If θ is an acute angle and $\sin\theta = \frac{1}{2019}$, then $\cos\theta$ can be written as $\frac{\sqrt{A}}{2019}$, where A is an integer. Calculate the sum of the digits of the remainder when $(A + 2019)$ is divided by 1000.

   (A) 18 (B) 19 (C) 20 (D) 21 (E) NOTA

3. What day will it be 2019 days from today, which is a Thursday?

   (A) Saturday (B) Sunday (C) Monday (D) Wednesday (E) NOTA

4. A 5×6 rectangle is revolved about one of its sides to produce a solid. If the sum of all possible volumes of such a solid can be written as $K\pi$, compute the sum of the digits of K.

   (A) 5 (B) 6 (C) 7 (D) 8 (E) NOTA

5. Compute the number of solutions to the equation:

   $$\sin 15x = \sin 7x \text{ where } 0 \le x < 2\pi$$

   (A) 7 (B) 10 (C) 18 (D) 30 (E) NOTA

6. Find the maximum value of $6\cos x - 5\tan x$.

(A) 1 (B) $\sqrt{11}$ (C) $\sqrt{61}$ (D) 11 (E) NOTA

7. X and Y are matrices such that

$$3X + Y = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 4 & 6 \\ 0 & 3 & 5 \end{bmatrix}$$
$$2X + 4Y = \begin{bmatrix} 9 & 0 & 5 \\ 3 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Calculate $\text{Trace}(X) + \text{Trace}(Y)$.

(A) 3 (B) 4 (C) 5 (D) 6 (E) NOTA

8. If the maximum value of $(\sin x)^4 + (\cos x)^4$ is $\frac{p}{q}$, where p and q are relatively prime positive integers, compute $p + q$.

(A) 8 (B) 10 (C) 58 (D) 92 (E) NOTA

9. We define a function $S_2(x)$ with the 2 rules below. Calculate $S_2(135436_8)$.
   1) If a number is binary, find the sum of its digits base 10. For example, $S_2(111_2) = 1 + 1 + 1 = 3$.
   2) If a number is not binary, convert it to binary, then carry out the first rule.

(A) 10 (B) 11 (C) 12 (D) 13 (E) NOTA

10. Sean and Jeremy are facing off in the Mathcounts countdown round. The format is as follows: a problem is placed on the screen, and each competitor has one attempt to solve the problem. Whoever gets the problem right first gets a point. If both miss the question, nobody gets a point. The winner is the first to 3 points. For Sean and Jeremy, the probability is 1 that one of them gets a point per question. Also, they are very evenly matched: the probability that they get a question right is split evenly.

    The stage is set and the game begins between the two. Unfortunately, the pressure gets to Sean, who rushes and turns in an incorrect answer for the first round. Jeremy, who is calm and collected, gets the question right half a minute later.

    Despite an early lead, Jeremy can still lose the match. Your goal is to calculate the probability that Sean wins, given these conditions.

    (A) 1/4 (B) 5/16 (C) 3/8 (D) 7/16 (E) NOTA

11. If the distance between the circumcenter and incenter of a 10-24-26 triangle can be written in the form $\sqrt{K}$, where K is an integer, find the sum of the digits of K.

    (A) 8 (B) 9 (C) 10 (D) 11 (E) NOTA

12. Let α and β be acute angles such that $\sin\alpha = \frac{3}{5}, \cos\beta = \frac{12}{13}$. $\cot(\alpha+\beta)$ can be expressed in the form $\frac{p}{q}$, where $|p|$ and q are relatively prime positive integers. Compute $p+q$.

    (A) 43 (B) 56 (C) 89 (D) 111 (E) NOTA

13. An equilateral triangle has area $48\sqrt{3}$. A point P lies inside the equilateral triangle. Let D, E, and F be the feet of the perpendiculars from P to sides AB, BC, and CA, respectively. If $PD = 7$ and $PE = 1$, compute PF.

    (A) 4 (B) 5 (C) $6\sqrt{3} - 3\sqrt{2}$ (D) Not enough info (E) NOTA

14. If $A^{-1} = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}, B^{-1} = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}$, and $(AB)^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, calculate $(a+b)-(c+d)$.

    (A) 10 (B) 15 (C) 20 (D) 25 (E) NOTA

15. Jenny makes friends easily. In fact, the number of friends Jenny has on Facebook is given by the relation 2^C, where C is the number of days after Jenny got her Facebook account. Let A be the smallest number of days after she got her Facebook account such that Jenny has 1,000,000 friends or more. (A is an integer.)
While Jenny is good at making friends, Jeffrey has a little trouble when it comes to making friends. He is, however, the subject of many memes. The number of memes made about Jeffrey is given by the relation

$$d_{n+3} = 3d_{n+2} + 8d_{n+1} + 8d_n$$

where d_n is the number of memes made about Jeffrey n days after Jeffrey started high school. Additionally, $d_0 = 0, d_1 = 2$, and $d_2 = 4$.
Compute the remainder when $d_{\left[\frac{A}{4}\right]}$ is divided by 10, where $[x]$ represents the greatest integer function.

(A) 2 (B) 4 (C) 6 (D) 8 (E) NOTA

16. Let the product of the solutions to the equation

$$x^2(\sin^{-1}(x^2) + \cos^{-1}(x^2)) = 1$$

be S. Calculate πS.

(A) -1 (B) 1 (C) $-\pi^2$ (D) π^2 (E) NOTA

17. Calculate the rank of the 2019×2019 matrix that has 2019 1s on the first row, followed by 2019 2s on the 2nd row, followed by 2019 3s on the third row, …, followed by 2019 2019s on the 2019th row.

(A) 0 (B) 1 (C) 2 (D) 2019 (E) NOTA

18. Let $P(x)$ be the polynomial of least degree with integral coefficients and leading coefficient 1 such that $P(x)$ has $1 + ki$ as a root, where k ranges over the integers from 1 to 2019. Let m be the largest positive integer such that $2^m | P(1)$. Compute m (note: $a|b$ means that a is a factor of b).

(A) 4022 (B) 4026 (C) 4030 (D) 4034 (E) NOTA

19. Which of the following best describes the polar graph $r = 2019$?

(A) Line (B) Cardiod (C) Rose Curve (D) Circle (E) NOTA

20. Given that $\sum_{n=0}^{\infty} \frac{n+1}{3^n}$ can be written in the form $\frac{A}{B}$ for relatively prime positive integers A, B, compute $A + B$.

(A) 5 (B) 9 (C) 13 (D) 17 (E) NOTA

21. Let X be the set $\{1^2, 2^3, 3^4, \ldots 2019^{2020}\}$.

If the number of nonempty subsets of X such that each subset consists only of elements whose units digit is 1 is $2^D - 1$, then compute D.

(A) 202 (B) 404 (C) 606 (D) 808 (E) NOTA

22. Calculate the number of integer pairs (x, y) that satisfy the equation:

$$x^2 + y^2 = 20192019$$

(A) 2 (B) 4 (C) 8 (D) 16 (E) NOTA

23. Find the number of real values of k such that $\begin{bmatrix} k & 2k & 0 \\ 3 & k & 1 \\ 0 & 3k & k \end{bmatrix}$ does not have an inverse.

(A)0 (B) 1 (C) 2 (D) 3 (E) NOTA

24. What is the range of $y = \arctan(x)$?

(A) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (B) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (C) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (D) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (E) NOTA

25. Calculate: $\sum_{n=1}^{\infty} \frac{1}{\sum_{k=1}^{n} k}$

(A) ln2 (B) e (C) 1 (D) 2 (E) NOTA

26. DC is touring the zoo to see all the wildlife. As the day goes on, the temperature begins to rise, and he decides to purchase an ice cream cone. Something doesn't look quite right, and while DC is staring at the cone, the vendor replies, "No need to worry. This cone can fit plenty of ice cream. Just take the points A(1,3,4), B(3,5,0), C(5,4,1), and D(0,3,6), and if you look at the parallelepiped with edges AB, AC, and AD, this cone has the same volume as that parallelepiped." DC is smart and realizes that the owner never gave units. He does not want to overpay for ice cream, so he moves on to find another vendor. Ignoring units, what was the volume of that ice cream cone?

(A) 6 (B) 8 (C) 10 (D) 12 (E) NOTA

27. Caroline is designing a shirt for the FAMAT state convention. Andy comes along, looks at the design, and says, "Look, that rose has n petals." Buffy hears the number that Andy says and replies, "Dude, that's not even possible." Andy replies, "Buffy, let's count them: 1, 2, 3, …, n." Buffy replies, "Oh, I thought you guys were talking about a rose curve of the form $a\cos(b\theta)$." Andy replies to this: "Dude…" Which of the following is a possible value for n?

(A) 3 (B) 6 (C) 12 (D) 24 (E) NOTA

For almost all of you, this will be the last MAO precalc competition test you will ever take. I hope next year proves to be a significant or integral year in your life. At any rate, the main division for next year is going to be Mu, so knowing calculus will come in handy. As this Gemini Test comes to a close (unless you skip around a lot like I used to ☺), I would like to provide a transition between Precalculus and Calculus, so the final few questions will explore concepts in calculus.

28. In calculus, one of the definitions of the derivative is:

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

With this in mind, calculate:

$$\frac{d}{dx}(x^2+4x+6)$$

(A) $2x+10$ (B) $2x+6$ (C) $2x+4$ (D) $2x$ (E) NOTA

29. One definition of the integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the x axis, and the lines $x = a, x = b$.

Given that $\int_0^{2019} 2019\, dx$ can be written as $2019K$ for some real number K, compute K.

(A) $\sqrt{2019}$ (B) 2019 (C) 2019^2 (D) 2019^3 (E) NOTA

30. In an isosceles triangle with legs of length 14 and base of length 8, a rectangle is inscribed. Given that the maximum area of such a rectangle can be written in the form $A\sqrt{B}$, where A, B are integers, and B is not divisible by the square of any prime, compute $A + B$.

(Note: this problem can be solved either with or without calculus.)

(A) 15 (B) 17 (C) 19 (D) 29 (E) NOTA