Alpha Gemini Solutions

1. Since we know it's a polynomial, let's first place a bound on the degree. As $101^2 = 10201 > 454$ we must have that the polynomial can be written in the form $ax + b$ Now notice $5(101) > 454$ so it follows, $0 \leq a \leq 4$. Thus our polynomials are $4x + 50, 3x + 151, 2x + 252, x + 353, 454$.Summing gives a total of $10x + 1260$ now plugging in 2019 we get a total of 21450 C
2. $\cos(x) = \sqrt{1 - (\sin(x))^2}$ where x is an acute angle. Now plugging in the givens we have $\cos\theta = \sqrt{\frac{2019^2-1}{2019}} \rightarrow A + B = 2019^2 + 2018 = 379 mod(1000)$
Summing the digits we get 19 B
3. Today is Thursday. 2019 leaves a remainder of 3 when divided by 7 so 2019 days from now it will be Sunday B
4. Notice the solid is going to be a cylinder and either the height is 5 or the radius is 5 so the sum of the volumes is

$$(5^2 \times 6 + 6^2 \times 5)\pi \rightarrow K = 330 \rightarrow \mathbf{B}$$

5. We have two cases, either the arguments differ by a multiple of 2π or the arguments sum to an odd multiple of π.

   Case 1: $15x = 7x + 2\pi k \rightarrow x = \frac{\pi k}{4} \rightarrow 0 \leq k < 8$ so we have 8 solutions here

   Case 2: $22x = (2k+1)\pi \rightarrow x = \frac{(2k+1)\pi}{22} \rightarrow 0 \leq k \leq 21$ so we have 22 solutions here. However note $\frac{\pi}{2}$ is double counted as it occurs in both cases. Summing we get a total of 30-1=29 solutions E

6. Notice as $x \rightarrow -\frac{3\pi}{2}$ the function grows to infinity E
7. First remark the property Trace(X) +Trace(Y)=Trace($X + Y$)
(See if you can prove it)
Add the two given equations to get:

$$5(X + Y) = \begin{pmatrix} 8 & 1 & 7 \\ 8 & 8 & 7 \\ 1 & 5 & 6 \end{pmatrix}$$

   Recalling that the trace is the sum of the elements of the main diagonal it follows the answer is 5 C
8. $a = \sin(x), b = \cos(x) \rightarrow (a^2 + b^2)^2 = a^4 + b^4 + 2a^2b^2 \rightarrow a^4 + b^4 = 1 - 2a^2b^2$
So maximum is 1 E
9. Although the first instinct when seeing a base problem is to convert to base 10, noticing that the problem is way too computational under the given time constraints motivates the idea to see if there is a trick involved. Indeed note that 8 is a power of 2 namely 2^3
Hence we can use the trick where one digit in base 8 corresponds to a matching three digits base 2. For example $56_8 = 101110_2$ (The first three digits come from the 5 and the next three come from the 6)

Using this we find the given number is equivalent to 1011101100011110_2 and so the answer is 10 **A**

10. We have a couple cases. One way Sean can win is if he wins the next three games consecutively which occurs with probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. The other way is if he loses again to Jeremy but manages to pull 3 wins. Thus in the next 4 games we could have JSSS, SJSS, and SSJS. As any arrangement of Ss and Js has equal probability, the probability in this case is $\frac{3}{2^4} = \frac{3}{16}$. Summing the two cases gives total probability $\frac{5}{16}$ B

11. Using the formula $A = rs$ we see the inradius of the triangle is 4 Now set the triangle on the coordinate plane. Let the right angle be at the origin and the other vertices be $(0,24)$ and $(10,0)$. We have the circumcenter to be the midpoint of the hypotenuse or $(5,12)$ and in the incenter is $(4,4)$ or (r, r) . Now using distance formula the distance is

$$\sqrt{65} \rightarrow \mathbf{D}$$

Note also the formula $D = \sqrt{R(r - 2r)}$ could have been used as well

12. We can just do $\tan(\alpha + \beta)$ and find the inverse. $\tan(\alpha) = \frac{3}{4}, \tan(\beta) = \frac{5}{13}$

$$\frac{\frac{3}{4} + \frac{5}{12}}{1 - (\frac{3}{4})(\frac{5}{12})} = \frac{56}{33} \rightarrow \mathbf{C}$$

13. We first calculate the side of the equilateral triangle to have length $8\sqrt{3}$. Now draw lines from P to A, B, C, respectively. Denote [X] to be the area of figure X.
Then notice $[PAC] + [PAB] + [PCB] = [ABC]$

$$4\sqrt{3}(7 + 1 + x) = 48\sqrt{3} \rightarrow x = 4 \rightarrow \mathbf{A}$$

14. $(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 12 \\ 5 & 8 \end{pmatrix} \rightarrow \mathbf{A}$

15. First we notice $A = 20$. This is not too difficult once we see that $2^{10} = 1024$, and $1000^2 = 1000000$
Hence we wish to compute d_5. Working mod 10 we have

$$d_{n+3} \text{mod} 10 = [3d_{n+2} + 8(d_{n+1} + d_n)] \text{mod} 10 = [3d_{n+2} - 2(d_{n+1} + d_n)] \text{mod} 10$$

Using the above we can just substitute the given values to arrive at 2 as the answer -> A

16. For $-1 \le a \le 1$ we have $\sin^{-1} a + \cos^{-1} a = \frac{\pi}{2}$

This can be seen by taking the sin of both sides of the equation and using sin addition (As an additional exercise see if you can carry out the proof ☺)

Then the equation simply becomes

$$x^2\left(\frac{\pi}{2}\right) = 1 \rightarrow x = \pm\sqrt{\frac{2}{\pi}}$$

Both solutions are defined so our answer is -2 -> E

17. Obviously if we pick a single element the determinant is non-zero. Hence if we can show that the rank is not 2 then obviously it is 1. Pick any 2 rows and any 2 elements from each row. Then the matrix is of the form $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$ which obviously has determinant 0
Thus the matrix has rank 1 -> B

18. Since the polynomial has integral coefficients we know $1 - ki$ is a root. Hence for a fixed value of k We can construct the polynomial

$$Q_k(x) = (x - 1 + ki)(x - 1 - ki) = (x^2 - 2x + k^2 + 1)$$

Notice that the polynomial we seek is $G(x) = Q_1(x)Q_2(x)Q_3(x) \ldots .Q_{2019}(x)$
Now observe that $Q_k(1) = k^2$ hence $G(1) = 1^2 2^2 3^2 \ldots .2019^2 = (2019!)^2$
Now to solve the problem it suffices to count the number of powers of 2 which divide 2019! Then double it

Using Legendre's this is just $\sum_{i=1}^{10} \left\lfloor \frac{2019}{2^i} \right\rfloor = 2011$ so the answer is 4022 **A**

19. $r^2 - 2019^2 \rightarrow x^2 + y^2 = 2019^2 \rightarrow$ **D**

20. $\sum_0^\infty \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}. A = \sum_0^\infty \frac{n}{3^n}, \frac{A}{3} = \sum_0^\infty \frac{n}{3^{n+1}} \rightarrow \frac{2A}{3} = \sum_1^\infty \frac{1}{3^n} \rightarrow A = \frac{3}{4}$

$\frac{3}{4} + \frac{3}{2} = \frac{9}{4}$ **C**

21. First we see that any number with units digit 1 will have units digit 1 after being raised to any nonnegative integral power so the valid numbers in this case are:1,11,21,20111-> 202 numbers

Similarly any number with units digit 9 will work in our case as any number with units digit 9 is odd so adding 1 to that number makes it even. But 9^{28} has units digit 1, hence the claim. This gives 202 numbers also

For units digits 3, and 7 we must have the exponent to be a multiple of 4 in order to produce a number with units digit 1. We know the exponent will either leave a remainder of 0 or 2 when divded by 4 so by symmetry we have 101 numbers that work for each of units digits 3 and units digit 7 so there are 202 numbers here

Summing we get 606 numbers and so D=606 -> **C**

22. Notice that $20192019 = 3 \bmod 4$. Notice however that $x^2 = 0,1 \bmod 4$. (This can be seen by testing 0,1,2,3. Hence the sum can never be 3. **E**

23. For a matrix to be non invertible the determinnat must be 0 One can see the determinant to be $k^3 - 9k^2$, so $k = 0,9$ are the solutions **C**

24. By definition arctan(x) lies in the first and fourth quadrants. The problem then reduces to determining the endpoints. We can work the problem in reverse. In particular $\tan(\frac{\pi}{2})$ is undefined which eliminates endpoints **A**

25 Using the formula for sum of integers which is $\frac{n(n+1)}{2}$ it remains to sum

$\frac{2}{(n)(n+1)} = \frac{2}{n} - \frac{2}{n+1}$ which telescopes to 2 **D**

26. We can calculate AB: <2,2,-4>, AC: <4,1,-3>, AD: <-1,0,2>

Thus we need to compute $\begin{vmatrix} 2 & 2 & -4 \\ 4 & 1 & -3 \\ -1 & 0 & 2 \end{vmatrix} = -10 \rightarrow |-10| = 10$ **C**

27. Any number divisible by 2 but not 4 could be n This is because if b odd the number of petals is b, which is odd. If b is even then the number of petals is 2b but be is even so 2b has 2 factors of 2 hence divisible by 4. The answer is 6 **B**

28. With direct substitution we find $f(x+h)-f(x)=4h+2xh+h^2$.

Divide by h to get

$$4+2x+h \to h=0 \to 2x+4 \to \mathbf{C}$$

29. Notice the area bound is simply a rectangle with height 2019 and width 2019 hence the answer desired is 2019 **B**

30.
The problem can be in fact generalized to an isosceles triangle with side a and base b.Notice that the top base of the rectangle splits the triangle into a smaller triangle and a trapezoidal figure. The smaller triangle is similar to the entire triangle. Call the base of the rectangle x and the height y, and let $h$be the height of the isosceles triangle

Now by similar triangles we have

$$\frac{h-y}{h}=\frac{x}{b} \to xh+by=bh \to (*)xh+by \geq 2\sqrt{xhby} \to xy \leq \frac{bh}{4}$$

Where (*) follows by AM-GM. Now plugging in the givens for our problem we have the maximum area to be $12\sqrt{5} \to \mathbf{B}$