For this test, unless otherwise specified:

- A deck of cards contains the standard 52 cards.
- Drawing $N$ objects is done at random, and without replacement.
- Multiple random numbers generated are independent.
- Random selection is done with uniform distribution (continuous or discrete).

For problems 1–2, $f(x) = (6 - x) - |6 - 2x|$. $X$ is a continuous random variable with probability density function (pdf) $p(X) = \begin{cases} af(X) & f(X) \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

1. Compute the value of $a$.

A. $\frac{1}{12}$ B. $\frac{1}{6}$ C. $\frac{1}{3}$ D. 1 E. NOTA

2. Compute the expected value of $X$.

A. $\frac{5}{3}$ B. 2 C. $\frac{7}{3}$ D. $\frac{8}{3}$ E. NOTA

3. The number of distinguishable permutations of the letters in PUZZLEHUNT is $\frac{11!}{n}$. Compute $n$.

A. 1 B. 2 C. 4 D. 8 E. NOTA

4. How many distinguishable permutations of the letters in ZIGZAG start and end with a consonant?

A. 36 B. 48 C. 60 D. 72 E. NOTA

5. A point $(x, y)$ is randomly selected with $|x| \leq 2$ and $0 \leq y \leq 4$. What is the expected number of tangent lines that can be drawn from this point to the graph of $y = x^2$?

A. $\frac{1}{6}$ B. $\frac{1}{3}$ C. $\frac{2}{3}$ D. $\frac{4}{3}$ E. NOTA

6. In the expansion of $(3x+y)^{24}$, the sum of the coefficients can be expressed as $a^b$ for integers $a, b$. How many ordered pairs $(a, b)$ are there?

A. 8 B. 10 C. 14 D. 18 E. NOTA

7. Compute the constant term in the expansion of $\left(x^2+1+\frac{1}{x}\right)^{10}$.

A. 102 B. 859 C. 3595 D. 4350 E. NOTA

8. Find the number of positive integers less than 6000 that are relatively prime to 50, but not to 36.

A. 200 B. 400 C. 800 D. 1600 E. NOTA

9. 12 points are equally spaced around a circle. 4 points are chosen to form a quadrilateral. How many incongruent quadrilaterals can be formed?

A. 15 B. 29 C. 35 D. 43 E. NOTA

10. A bag contains 3 red, 4 green, and 5 blue marbles. Three marbles are drawn at random. What is the probability that all three are of different colors?

A. $\frac{5}{144}$ B. $\frac{1}{22}$ C. $\frac{5}{24}$ D. $\frac{3}{11}$ E. NOTA

11. Consider the polar graph $r=1+2\cos\theta$. If a point is randomly chosen from within the outer loop of the limacon, what is the probability that the point is within the inner loop?

A. $\frac{2\pi-3\sqrt{3}}{4\pi+6\sqrt{3}}$ B. $\frac{2\pi-3\sqrt{3}}{4\pi+3\sqrt{3}}$ C. $\frac{2\pi-3\sqrt{3}}{2\pi+6\sqrt{3}}$ D. $\frac{2\pi-3\sqrt{3}}{2\pi+3\sqrt{3}}$ E. NOTA

12. Five cards are drawn from a standard deck. The probability of drawing a full house can be expressed as $\frac{k}{\binom{52}{5}}$. Compute $k$. (A full house consists of three of a kind and a pair: for example, A-A-A-K-K.)

A. 1248 B. 1352 C. 3744 D. 4056 E. NOTA

13. A drawer contains $a$ identical red socks and $b$ identical white socks for positive even integers $a, b$. If two socks are drawn at random from the drawer, the probability of drawing a pair of matching socks is exactly 0.5. Find the third smallest possible value of $a + b$.

A. 16 B. 36 C. 64 D. 100 E. NOTA

14. $\int_0^4 x^3\, dx$ is approximated using a Riemann sum with four equal subintervals. In each subinterval, a random $x$ value is selected, and the corresponding $y$ value is used as the height. What is the expected result of the described Riemann sum?

A. 62 B. 64 C. 66 D. 68 E. NOTA

15. 50 students are asked about their science class at school. Half of the students take Physics, and 30 take Chemistry. There are 10 students who take neither. If a randomly selected student takes Physics or Chemistry, what is the probability that the student takes both?

A. $\frac{1}{10}$ B. $\frac{1}{8}$ C. $\frac{3}{10}$ D. $\frac{3}{8}$ E. NOTA

16. In the expansion of $(x + 2)^{31}$, what is the degree of the term with the greatest coefficient?

A. 0 B. 10 C. 11 D. 16 E. NOTA

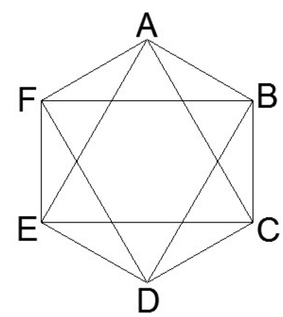
17. Rectangle $ABCD$ has area of 120. $E$ is a randomly selected point on $\overline{CD}$. If $\overline{AC}$ and $\overline{BD}$ intersect at $M$, and $\overline{AE}$ and $\overline{BD}$ intersect at $N$, find the expected area enclosed in $\Delta MAN$.

A. 10 B. 15 C. $60\ln 2 - 30$ D. $60 - 60\ln 2$ E. NOTA

18. Find the coefficient of the $x^2y^3z^2$ term in the expansion of $(x+y+z)^7$.

A. 210 B. 350 C. 441 D. 735 E. NOTA

For 19–21, consider the graph on the right, which consists of 6 vertices and 12 edges, connecting all pairs of vertices that are not opposite each other. A **path** is a sequence of connected edges denoted by the vertices visited in order from start to end. The **length** of a path is the number of edges the path contains. A **circuit** is a path that starts and ends at the same vertex. For example, $A - C - E - A$ is a circuit of length 3 that traces out $\Delta ACE$ from vertex $A$ to vertex $A$.




19. Find the number of paths from $A$ to $D$, if no vertex may be visited more than once.

A. 4 B. 8 C. 16 D. 32 E. NOTA

20. Find the number of paths from $A$ to $D$ with length 4.

A. 16 B. 32 C. 48 D. 64 E. NOTA

21. Find the number of circuits that visit each vertex (other than the starting/ending vertex) exactly once.

A. 48 B. 96 C. 144 D. 192 E. NOTA

22. The region bounded between the graphs of $y = x^2$ and $x = y^3$ is revolved around the vertical line $x = k$, where $k$ is randomly selected on the interval $[-6, 0]$. Find the expected volume of revolution.

A. $\frac{25\pi}{21}$ B. $\frac{85\pi}{42}$ C. $\frac{20\pi}{7}$ D. $\frac{155\pi}{42}$ E. NOTA

23. Consider the graph of $f(x) = \frac{2}{3}x^{\frac{3}{2}}$ on the interval $[0, 8]$. A point is chosen at random along the curve, and line $\ell$ is drawn tangent to $f(x)$ at that point. What is the probability that the $x$-intercept of $\ell$ is greater than 1?

A. $\frac{7}{26}$ B. $\frac{3}{8}$ C. $\frac{5}{8}$ D. $\frac{19}{26}$ E. NOTA

24. When all the distinguishable permutations of the letters in VEGAS are written in alphabetical order, find the 60th in the list.

A. GESVA B. GEVSA C. GSAEV D. GVAES E. NOTA

25. Let $f(x) = ax^3 + bx^2 + cx + d$. Each of $a, b, c, d$ is randomly selected from the set $\{-2, -1, 0, 1, 2\}$ with replacement. What is the probability that $f(x) = 0$ has no solutions over the real numbers?

A. $\frac{26}{625}$ B. $\frac{36}{625}$ C. $\frac{8}{125}$ D. $\frac{36}{125}$ E. NOTA

26. Sequences of 6 letters are made using only (but not necessarily all of) A, B, C, and D. How many different sequences contain at least one pair of consecutive letters that are the same?

A. 1620 B. 3072 C. 3124 D. 3367 E. NOTA

27. Three points are chosen at random on a circle of radius 1. What is the probability that an acute triangle is formed when they are connected?

A. $\frac{1}{4}$ B. $\frac{1}{\pi}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$ E. NOTA

28. Three points are chosen at random on a circle of radius 1. If an acute triangle is formed when they are connected, what is the expected area enclosed by the triangle?

A. $\frac{1}{\pi}$ B. $\frac{3\sqrt{3}}{8}$ C. $\frac{3}{\pi}$ D. $\frac{\pi}{2}$ E. NOTA

Quick sort is an algorithm for sorting a collection sortable items (say numbers). In each step, an element is chosen as a pivot, and the collection is partitioned in two – one set smaller than the pivot, and rest greater than or equal to the pivot. Quick sort is then performed on the partitions. Ideally, the median is chosen so that the two halves are equal in size. However, this is not practical, as finding the median requires the collection to be sorted.

In 29 and 30, we will examine two simple schemes. Asymptotically, we can treat the collection of elements as infinitely sized, with each number in $[0,1]$ representing the location of an element after sorting. Randomly selecting an element is equivalent to randomly selecting a number on $[0,1]$, with 0.5 being the median.

29. If one random element is selected as the pivot, what is its expected distance from the median, measured on the $[0,1]$ scale?

A. $\frac{1}{6}$ B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{1}{e}$ E. NOTA

30. In a median-of-three scheme, three elements are selected at random, and the median of the three elements is chosen as the pivot. What is its expected distance from the median, measured on the $[0,1]$ scale?

A. $\frac{1}{32}$ B. $\frac{1}{16}$ C. $\frac{3}{16}$ D. $\frac{3}{8}$ E. NOTA