1. B The positive section of $f(x)$ consists of two segments, connection $(0,0)$ to $(3,3)$ to $(4,0)$. The area under any probability density function must be 1. However, the area under the positive section of $f(x)$ is $\frac{1}{2} \cdot 4 \cdot 3 = 6$, so $a = \frac{1}{6}$.

2. C $E(x) = \int_0^4 xp(x)\,dx = \frac{1}{6}\left(\int_0^3 x^2\,dx + \int_3^4 x(12-3x)\,dx\right)$
$= \frac{1}{6}\left(\left[\frac{1}{3}x^3\right]_0^3 + \left[\frac{12\ \ ^2}{2} - \frac{3x^3}{3}\right]_3^4\right) = \frac{1}{6}\left(9 + (96-64) - (54-27)\right) = \frac{7}{3}$

3. E $\frac{10!}{2!2!} = \frac{11!}{44}$, so $n = 44$.

4. D Pretending the word consist of 4 distinct consonants and 2 distinct vowels, the number of arrangements would be $4 \cdot 3 \cdot 4!$. However, with 2 each of Z and G, there are four copies of each arrangement. So the number of distinguishable ones is 72.

5. C If the point is selected above $y = x^2$, no tangent line can be drawn. If the point is selected below, two tangent lines can be drawn. The region has an area of 16. With an area of $2\int_0^2 x^2\,dx = 2\left(\left[\frac{x^3}{3}\right]_0^2\right) = \frac{16}{3}$. So the expected value is $0 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{2}{3}$.

6. D The sum of coefficients is $4^{24} = 2^{48}$. So as long as $b > 0$ and $b|48$, there is a corresponding value for a. $48 = 2^4 \cdot 3$, so it has 10 factors, only two of which are odd. If b is even, there are two values of a (positive and negative), and if b is odd, there is only one. So there are a total of $2 \cdot 8 + 1 \cdot 2 = 18$ ordered pairs.

7. E There are a few ways to create the constant term, arranged in decreasing powers of 1:
$1^{10} + \frac{10!}{1!7!2!}(x^2)^1(1)^7\left(\frac{1}{x}\right)^2 + \frac{10!}{2!4!4!}(x^2)^2(1)^4\left(\frac{1}{x}\right)^4 + \frac{10!}{3!1!6!}(x^2)^3(1)^1\left(\frac{1}{x}\right)^6 = 4351.$

8. C An integer is relatively prime to 50 but not 36 if it is a multiple of 3 but not of 2 or 5. So of 6000 integers, $\frac{1}{3}$ of which are multiples of 3, $\frac{1}{2}$ of which are not multiples of 2, and $\frac{4}{5}$ of which are not multiples of 5. In total, there are $6000\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{4}{5}\right) = 800$ numbers satisfying the condition.

9. B Consider the gaps between consecutive vertices. For instance, a gap of 1 means the two vertices are adjacent, where a gap of 6 means the two points are diametrically opposed. To count the number of incongruent quadrilaterals, we need to consider all partitions of 12 into 4 positive integers. The partitions are: 9111, 8211, 7311, 7221, 6411, 6321, 6222, 5511, 5421, 5331, 5322, 4431, 4422, 4332, 3333.
If a partition is of the form AAAA or AAAB, there is only one possible quadrilateral. There are 3 such partitions, for 3 quadrilaterals. If a partition is of the form AABB or AABC, there are 2 possible quadrilaterals, where the two sides of gap A can be adjacent or opposite. There are 10 such partitions, for 20 quadrilaterals. If a partition is of the form ABCD, there are 3 possible quadrilaterals, where the opposite of A can be B, C, or D. There are 2 such partitions, for a total of 6 quadrilaterals. In total, there are 29 possible quadrilaterals.

10. D The probability is $\frac{3\cdot 4\cdot 5}{\binom{12}{3}} = \frac{3}{11}$.

11. B As θ ranges from 0 to 2π,$r = 0$ when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. The probability is the area of the inner loop over the area of the outer loop.
The area of the outer loop is
$2 \cdot \frac{1}{2}\int_0^{\frac{2\pi}{3}}(1 + 2\cos\theta)^2\, d\theta = \int_0^{\frac{2\pi}{3}}(1 + 4\cos\theta + 4\cos^2\theta)\, d\theta$
$= \int_0^{\frac{2\pi}{3}}\big(1 + 4\cos\theta + 2(\cos 2\theta + 1)\big)\, d\theta = [3\theta + 4\sin\theta + \sin 2\theta]_0^{\frac{2\pi}{3}} = 2\pi + \frac{3\sqrt{3}}{2}$
The area of the inner loop is
$2 \cdot \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}(1 + 2\cos\theta)^2\, d\theta = [3\theta + 4\sin\theta + \sin 2\theta]_{\frac{2\pi}{3}}^{\pi} = \pi - \frac{3\sqrt{3}}{2}$
Finally, the probability is $\frac{2\pi - 3\sqrt{3}}{4\pi + 3\sqrt{3}}$

12. C There are 13 ways to select the value of the three of a kind, and 4 ways to choose their suits. There are 12 ways to select the value of the pair, and 6 ways to choose their suits. There are $13 \cdot 4 \cdot 12 \cdot 6 = 3744$ different full houses.

13. E The probability of drawing a pair of matching socks is $\frac{a(a-1)+b(b-1)}{(a+b)(a+b-1)} = \frac{1}{2}$.
Expanding to get $2a^2 - 2a + 2b^2 - 2b = a^2 + 2ab + b^2 - a - b$,
Collecting to get $a^2 - 2ab + b^2 - a - b = 0$, or $(a-b)^2 = a + b$.
In other words, $a + b$ must be a perfect square. Let $n = a - b$, we can consider the system $a + b = n^2$, $a - b = n$, which has solutions $a = \frac{n^2+n}{2}, b = \frac{n^2-n}{2}$. Since a, b are even integers, it follows that n must be a multiple of 4. The third smallest possible value of n then is 12, making $a + b = 144$.

14. B Consider any interval $[a, b]$, if a value of x is selected at random, then the expected height is $\frac{1}{b-a}\int_a^b x^3\,dx$, and the corresponding rectangular approximation for the interval would have an expected area of $\int_a^b x^3\,dx$. In other words, a Riemann sum approximation done in this manner simply has expected area of $\int_0^4 x^3\,dx = 64$.

15. D There are 40 students taking at least one of the two classes, and $25 + 30 - 40 = 15$ students taking both classes. So the probability is $\frac{15}{40} = \frac{3}{8}$.

16. B Consider consecutive terms in the expansion, they would look like $\binom{31}{n} x^n 2^{31-n}$ and $\binom{31}{n-1} x^{n-1} 2^{31-n+1}$. So the ratio of coefficients is $\frac{\binom{31}{n-1}2^{31-n+}}{\binom{31}{n}2^{31-n}} = \frac{2n}{32-n}$. We simply need to determine the last value of n such that $\frac{2n}{32-n} > 1$, indicating an increase in coefficient. This occurs at $n = 11$, indicating an increasing exponent from x^{11} term to x^{10} term, and x^{10} term has the greatest coefficient.

17. C Let $\frac{DE}{DC} = x$, then x ranges from 0 to 1, as E moves from D to C.
$\Delta DEN \sim \Delta BAN$, so $\frac{DN}{BN} = x$, and $\frac{MN}{BD} = \frac{\frac{1+x}{2} - x}{1+x} = \frac{1-x}{2+2x}$. Since ΔMAN and ΔBAD share a height from vertex A, the ratio of their areas is simply a ratio of the bases MN and BD. $[BAD] = \frac{1}{2}[ABCD] = 60$, so $[MAN] = 30\left(\frac{1-x}{1+x}\right)$.
The expected area is then $30\int_0^1 \frac{1-x}{1+x}\,dx = 30\int_0^1 \left(-1 + \frac{2}{1+x}\right)dx$
$= 30[-x + 2\ln(1+x)]_0^1 = 30(2\ln 2 - 1)$

18. A $\frac{7!}{2!3!2!} = 210$

19. E In general, the graph is isomorphic to the vertices and edges of an octahedron, which may be a helpful way of visualizing the paths. For this question in particular, the first edge must lead from A to one of B, C, E, F. All four of those vertices are in equivalent positions with respect to A and D, so without loss of generality, let the first edge be AB. From B, the next vertex can be D, C, F. D ends the path. We can consider traveling to C, F as traveling clockwise or counterclockwise around rectangle $BCEF$, and this can end after 1, 2, or 3 additional vertices to B by traveling to D. So there are 7 total paths from B to D, which means $4 \cdot 7 = 28$ paths from A to D.

20. C We will consider the paths of length 4 from A to D by whether they visit A or D in the middle of the path. There are a few cases: $AXYZD, AXAYD, AXDYD$, where X, Y, Z each represent one of B, C, E, F, not necessarily distinct. In the first case, there are 4 ways to select X, and 2 each to select Y, Z. In the other cases, there are 4 ways each to select X and Y. This makes for a total of 48 paths.

21. D Without the loss of generality, we will consider the first path as AB. Then we can compute the number of circuits by when vertex D is reached. The cases are $ABDXYZA, ABXDYZA, ABXYDZA$, where X, Y, Z each represent one of C, E, F. This time, they do have to be distinct. Note that D cannot be the 6th vertex reached, as there would be no path leading back to A.
In the first case, there are 2 ways to select XYZ, specifically CEF, FEC. In the second case, there are 2 ways to select X, and 2 ways to select YZ. In the third case, there are 2 ways to select XY, and 1 way to select Z. In total, there are 8 circuits starting with AB, and 24 ways to select equivalence of AB, for 192 circuits.

22. C By Pappus's Theorem, the $V = 2\pi RA$, where R is the distance from centroid of the region to the axis of revolution, and A is the area of the region. Thus the expected volume is $\int_{-6}^{0} 2\pi(x_0 - x)A \cdot \frac{1}{6}\,dx$, where x_0 is the x-coordinate of the centroid. Since every quantity except $(x_0 - x)$ is a constant in the integral, the expected volume can be rewritten as $2\pi A \int_{-6}^{0}(x_0 - x) \cdot \frac{1}{6}\,dx$. $\int_{-6}^{0}(x_0 - x) \cdot \frac{1}{6}\,dx$ is the expected distance from centroid to axis of revolution, which occurs when the axis is located at $x = -3$. Thus it is sufficient to compute the volume of revolution when the axis is at $x = -3$, which is

$$2\pi \int_0^1 (x+3)\left(x^{\frac{1}{3}} - x^2\right) dx = 2\pi \int_0^1 \left(-x^3 - 3x^2 + x^{\frac{4}{3}} + 3x^{\frac{1}{3}}\right) dx = \frac{20\pi}{7}$$

23. D First, we need to determine the point on the graph where the tangent line has an x-intercept of 1. Call that point $\left(x, \frac{2}{3}x^{\frac{3}{2}}\right)$, the slope of the tangent can be computed in two ways – $f'(x) = x^{\frac{1}{2}}$, and slope formula to $(1, 0)$. Thus $\frac{\frac{2}{3}x^{\frac{3}{2}} - 0}{x - 1} = x^{\frac{1}{2}}$. Solving to get $x = 0, 3$. When $x > 3$, the x-intercept will be greater than 1.
However, the point is chosen on the curve, so the probability is the ratio of the arclength of the graph from 3 to 8 to the arclength from 0 to 8. Arclength from 3 to 8 can is $\int_3^8 \sqrt{1 + \left(f'(x)\right)^2}\,dx = \int_3^8 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_3^8 = \frac{2}{3}(27 - 8) = \frac{38}{3}$.
Similarly, the arclength from 0 to 8 is $\frac{2}{3}(27 - 1) = \frac{52}{3}$. So the probability is $\frac{38}{52} = \frac{19}{26}$.

24. B There are 24 permutations each starting with A and E. Then there are 6 each starting with GA and GE. The 60th permutation is the last one starting with GE, which is GEVSA.

25. C For $f(x) = 0$ to have no solution, a is necessarily 0. Next, $bx^2 + cx + d$ must have a negative discriminant, or $c^2 - 4bd < 0$. This cannot occur unless b, d are both positive or both negative. It is then simple enough to consider the 3 cases of $4bd$. $4bd$ can be 4, 8, or 16, with probabilities of $\frac{2}{25}, \frac{4}{25}$, and $\frac{2}{25}$ respectively. When $4bd = 4$, $c \neq \pm 2$, or there would be a real solution. So $bx^2 + cx + d$ has no real solution with probability $\frac{2}{25} \cdot \frac{3}{5} + \frac{6}{25} = \frac{36}{125}$. Add in the restriction of $a = 0$ for a probability of $\frac{36}{625}$. However, the constant functions that slip through as they technically have a discriminant of 0 when treated as a quadratic. Four of the five constant functions do not have a solution, making the final probability $\frac{40}{625} = \frac{8}{125}$.

26. C It is easier to compute this by complement. There are 4^6 6-letter sequences, of those $4 \cdot 3^5$ do not contain any pair of consecutive letters that match. $4^6 - 4 \cdot 3^5 = 4(1024 - 243) = 3124$.

27. A Call the 3 points chosen A, B, C, and place the circle centered at the origin, which we will call O. Without loss of generality, place A at $(1, 0)$ by rotating the figure. Place B in quadrants I or II by reflecting the figure, if necessary. C has no restrictions when selected. Draw diameters through A and B. For ΔABC to be acute, C must be on the sector opposite of minor arc AB. Otherwise, one of the three central angles is greater than π, causing the corresponding inscribed angle to be obtuse.
Let's call $\angle AOB = x$, then x is selected at random on $[0, \pi]$. Let y be $\angle AOC$, but measured clockwise, rather than strictly the smallest angle between OA and OC. Then y is selected at random on $[0, 2\pi]$. ΔABC is acute if $\pi - x < y < \pi$.
We can now compute the probability geometrically. x, y are selected from a rectangle of dimensions π by 2π. ΔABC is acute if the point is below $y = \pi$ and above $y = \pi - x$, which compose of $\frac{1}{4}$ of the rectangle.

28. C The area of the valid region described in #27 has an area of $\frac{\pi^2}{2}$, so each point within has a density of $\frac{2}{\pi^2}$.
Next, we can compute the area of ΔABC in terms of x, y. $[AOB] = \frac{1}{2}\sin x$, $[AOC] = \frac{1}{2}\sin y$, and $[BOC] = \frac{1}{2}\sin(2\pi - x - y) = -\frac{1}{2}\sin(x + y)$.
Therefore, $[ABC] = \frac{1}{2}(\sin x + \sin y - \sin(x + y))$.
The expected area of ΔABC is then $\frac{2}{\pi^2} \cdot \frac{1}{2}\int_0^{\pi}\int_{\pi-x}^{\pi}(\sin x + \sin y - \sin(x + y))dy\,dx$
Integrating the inner layer to get $[y\sin x - \cos y + \cos(x + y)]_{\pi-x}^{\pi}$.
Working with each part:
$[y\sin x]_{\pi-x}^{\pi} = \sin x\left(\pi - (\pi - x)\right) = x\sin x$
$[-\cos y]_{\pi-x}^{\pi} = -\cos\pi + \cos(\pi - x) = 1 - \cos x$
$[\cos(x + y)]_{\pi-x}^{\pi} = \cos(x + \pi) - \cos(x + \pi - x) = -\cos x + 1$

Combining and placing it back in the outer layer to get:
$\frac{1}{\pi^2}\int_0^{\pi}(x\sin x + 2 - 2\cos x)\,dx = \frac{1}{\pi^2}[-x\cos x + \sin x + 2x - 2\sin x]_0^{\pi} = \frac{3\pi}{\pi^2} = \frac{3}{\pi}$.

29. B A randomly selected element is equally likely to be above or below the median. In each half, the expected selection is in the middle, at $\frac{1}{4}$ or $\frac{3}{4}$. Both are $\frac{1}{4}$ away from the median.

30. C Let the 3 elements chosen be x, y, z, each on $[0, 1]$, so the sample space is a unit cube. Without loss of generality, let $x < y < z$. Since there are 3! =6 ordering of x, y, z, the selected case occupies a volume of $\frac{1}{6}$. We can simply treat the problem as selecting from that section, with each point (x, y, z) having a density of 6. y is the median of three, and its distance from the actual median is $\left|y - \frac{1}{2}\right|$.
The expected distance is $6\int_0^1\int_x^1\int_y^1\left|y-\frac{1}{2}\right|dz\,dy\,dx = 6\int_0^1\int_x^1\left|y-\frac{1}{2}\right|(1-y)\,dy\,dx$
To compute the next layer, it is necessary to consider whether y is above or below $\frac{1}{2}$. Consequently, we must also break down the outer layer to whether x is above or below $\frac{1}{2}$. This makes for 3 cases:
$\int_{\frac{1}{2}}^1\int_x^1\left(y-\frac{1}{2}\right)(1-y)\,dy\,dx = \frac{1}{192}$
$\int_0^{\frac{1}{2}}\int_x^{\frac{1}{2}}\left(\frac{1}{2}-y\right)(1-y)\,dy\,dx = \frac{1}{64}$
$\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left(y-\frac{1}{2}\right)(1-y)\,dy\,dx = \frac{1}{96}$
Combining to get the expected value as $6\left(\frac{1}{192}+\frac{3}{192}+\frac{2}{192}\right) = \frac{36}{192} = \frac{3}{16}$.