1) Find $y(2)$ if $y(x)$ is a positive function that satisfies the differential equation $xy' = y + 1$ and $y(1) = 2018$.

(A) 2018 (B) 4037

(C) 2020 (D) 4039 (E) NOTA

2) Which of the following relations satisfies the differential equation $(x + y)y' = x - y$ with $y = 0$ when $x = 1$?

(A) $x^2 - 2xy - y^2 = 1$ (B) $x^2 - x - y = 0$

(C) $x^2 - 2xy - y^2 - x^4 = 0$ (D) $x^2 - 2xy - y^2 - x = 0$ (E) NOTA

3) If $y' + 4x^3y = x^3$ and $y = 2$ when $x = 0$, what is y when $x = 1$?

(A) $\frac{4e+7}{4e}$ (B) $\frac{4e+5}{4e}$

(C) $\frac{e+7}{4e}$ (D) $\frac{e+5}{4e}$ (E) NOTA

4) If $y' + \frac{1}{x}y = y^3$, $y = 2$ when $x = 1$, and $y > 0$, what y when $x = \frac{1}{2}$?

(A) $\frac{16}{23}$ (B) $\frac{16}{9}$

(C) $\frac{4\sqrt{23}}{23}$ (D) $\frac{4}{3}$ (E) NOTA

5) Find $y(2)$ if $y(x) > 0$ is the solution to the differential equation $y' = \frac{y}{x}$ and $y(1) = 2019$.

(A) -2019 (B) 2017

(C) 2019 (D) -2017 (E) NOTA

6) Given that $M(x, y) + y\sec(x)\frac{dy}{dx} = 0$ is an exact differential equation, which of the following is a possible value of $M(x, y)$?

(A) $y\ln|\sec(x) + \tan(x)|$ (B) $y^2\sec^2(x)$

(C) $\frac{1}{2}y^2\sec(x)\tan(x) + \sec^2(x)$ (D) $\frac{1}{2}y^2\sec(x)$ (E) NOTA

7) Using the correct answer to Question 6, solve the exact equation in Question 6 given the additional information that $y = 0$ when $x = 0$. Which of the following is a possible value of x when $y = 1$?

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$ (D) $\frac{7\pi}{6}$ (E) NOTA

8) What is the general form of the solution to $y''' - y'' - 9y' + 9y = 0$? Assume y is a function of x.

(A) $y = C_1e^{-3x} + C_2e^{x} + C_3e^{3x}$ (B) $y = C_1e^{-x} + C_2e^{x} + C_3e^{3x}$

(C) $y = C_1e^{-3x} + C_2e^{-x} + C_3e^{3x}$ (D) $y = C_1e^{-9x} + C_2e^{x}$ (E) NOTA

9) What is the general form of the solution to $y'' + 4y = 0$? Assume y is a function of x.

(A) $y = C_1e^{-2x} + C_2e^{2x}$ (B) $y = C_1\cos(2x) + C_2\sin(2x)$

(C) $y = C_1e^{-4x} + C_2e^{4x}$ (D) $y = C_1\cos(4x) + C_2\sin(4x)$ (E) NOTA

10) What is the general form of the solution to $y'' + 4y' + 4y = 0$? Assume y is a function of x.

(A) $y = C_1e^{-2x} + C_2e^{-2x}$ (B) $y = C_1e^{-2x} + C_2x^{-2}$

(C) $y = C_1e^{-2x} + C_2e^{2x}$ (D) $y = C_1e^{-2x} + C_2xe^{-2x}$ (E) NOTA

11) Of the six functions listed below, which of the following sets of three will not result in a Wronskian of zero?

I. $y = \sin^2(2x)$ II. $y = \cos^2(2x)$ III. $y = \sin(4x)$

IV. $y = \sin(2x)$ V. $y = \cos(4x)$ VI. $y = 2$

(A) I, II, and VI (B) II, V, and VI

(C) III, IV, and V (D) I, V, and VI (E) NOTA

12) A tank has pure water flowing into it at 12 liters per minute. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liters per minute. Initially, the tank contains 10 kg of salt in 100 liters of water. If the tank can hold at most 1,000 liters of water, what will the amount of salt (in kg) in the tank be when the tank is full?

(A) $\frac{1}{100{,}000}$ (B) $\frac{1}{10{,}000}$

(C) $\frac{1}{1{,}000}$ (D) $\frac{1}{100}$ (E) NOTA

13) A cat starts at the origin and runs with a speed of 2 along the positive y-axis in the positive direction. A dog starts at the point $(9,0)$ and runs with a speed of 4, always in the direction of the instantaneous location of the cat. The graph of which of the following equations coincides with the curve traced by the path of the dog?

(A) $y = -3\sqrt{x} + \frac{1}{9}x^{\frac{3}{2}} + 6$ (B) $y = 6\sqrt{x} - \frac{1}{9}x^{\frac{3}{2}} - 15$

(C) $y = 3\sqrt{x} - \frac{1}{9}x^{-\frac{3}{2}} - \frac{728}{81}$ (D) $y = 6\sqrt{x} - \frac{1}{9}x^{-\frac{3}{2}} - \frac{1457}{81}$ (E) NOTA

In problems 14-16 below, assume that the general solution can be written as $y = y_h + y_p$, where y_h is the general solution to the homogenous version of linear equation and y_p is referred to as the *particular solution*.

14) What is the particular solution of $y''' - y'' - 9y' - 9y = x^2$? Assume y is a function of x.

(A) $y_p = \frac{1}{9}x^2 - \frac{2}{9}x + \frac{7}{81}$ (B) $y_p = -\frac{1}{9}x^2 - \frac{2}{9}x + \frac{7}{81}$

(C) $y_p = -\frac{1}{9}x^2 + \frac{2}{9}x - \frac{16}{81}$ (D) $y_p = \frac{1}{9}x^2 - \frac{2}{9}x - \frac{7}{81}$ (E) NOTA

15) What is the particular solution of $y'' + 4y = \cos(2x)$? Assume y is a function of x.

(A) $y_p = \frac{1}{4}x\sin(2x)$ (B) $y_p = \frac{1}{4}\sin(2x)$

(C) $y_p = \frac{1}{4}x\cos(2x)$ (D) $y_p = \frac{1}{4}\cos(2x)$ (E) NOTA

16) What is the particular solution of $y'' + 4y' + 4y = \sqrt[3]{x}e^{-2x}$? Assume y is a function of x.

(A) $-\frac{9}{28}x^{\frac{5}{3}}e^{-2x}$ (B) $-\frac{9}{28}x^{\frac{7}{3}}e^{-2x}$

(C) $\frac{9}{28}x^{\frac{5}{3}}e^{-2x}$ (D) $\frac{9}{28}x^{\frac{7}{3}}e^{-2x}$ (E) NOTA

17) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be a solution to the differential equation $y'' + xy = e^x$ with $y(0) = y'(0) = 1$. Which of the following is a recursion relation satisfied by the sequence of coefficients $\{a_n\}$ for $n > 0$?

(A) $a_{n+2} = \frac{1}{n!} - \frac{a_n}{(n+2)(n+1)}$ (B) $a_{n+2} = \frac{1}{n!} - \frac{a_{n-1}}{(n+2)(n+1)}$

(C) $a_{n+2} = \frac{1}{(n+2)!} - \frac{a_n}{(n+2)(n+1)}$ (D) $a_{n+2} = \frac{1}{(n+2)!} - \frac{a_{n-1}}{(n+2)(n+1)}$ (E) NOTA

18) Consider the series solution to $y'' + xy = e^x$ with $y(0) = y'(0) = 1$ as described in Question 17 above. Find $a_0 + a_1 + a_2 + a_3 + a_4 + a_5$.

(A) $\frac{173}{120}$ (B) $\frac{35}{24}$

(C) $\frac{293}{120}$ (D) $\frac{59}{24}$ (E) NOTA

19) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be a solution to the differential equation $(x^2 - 2x + 2)y'' + xy = 0$ with $y(0) = y'(0) = 1$. Find the radius of convergence of this series.

(A) 1 (B) $\sqrt{2}$

(C) 2 (D) ∞ (E) NOTA

20) What are the eigenvalues of $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ -4 & 0 & 1 \end{bmatrix}$?

(A) $\{1, 0, -3\}$ (B) $\{-1, 0, 3\}$

(C) $\{1, 0, 3\}$ (D) $\{-1, 0, -3\}$ (E) NOTA

21) What are the eigenvectors of $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ -4 & 0 & 1 \end{bmatrix}$?

(A) $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ (D) $\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ (E) NOTA

22) If $x(t), y(t)$, and $z(t)$ are continuous differentiable functions satisfying the equations $\begin{bmatrix} x' = x - z \\ y' = 2x + z \\ z' = -4x + z \end{bmatrix}$ then what is the most general solution vector $\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$?

(A) $\begin{bmatrix} c_1e^{t} + c_3e^{-3t} \\ c_2 \\ 2c_1e^{t} - 2c_3e^{-3t} \end{bmatrix}$ (B) $\begin{bmatrix} c_1e^{-t} + c_3e^{3t} \\ c_2 \\ 2c_1e^{-t} - 2c_3e^{3t} \end{bmatrix}$

(C) $\begin{bmatrix} c_1e^{t} + c_3e^{-3t} \\ -4c_1e^{t} + c_2 \\ 2c_1e^{t} - 2c_3e^{-3t} \end{bmatrix}$ (D) $\begin{bmatrix} c_1e^{-t} + c_3e^{3t} \\ -4c_1e^{-t} + c_2 \\ 2c_1e^{-t} - 2c_3e^{3t} \end{bmatrix}$ (E) NOTA

23) Assume that a solution of the initial value problem $y'' = x + y - y^2, y(0) = -1, y'(0) = 1$ exists and has the form $y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$

What is the sum of the first five coefficients of this Taylor expansion?

A) -1 B) 1/2

C) -2/3 D) 4/3 E) NOTA

24) What is the general form of the solution to $y' = 2x + \frac{1}{x}$?

(A) $y = x^2 - \frac{1}{x^2}$ (B) $y = x^2 + \ln(x)$

(C) $y = 2 - \frac{1}{x^2}$ (D) $y = 2 + \ln(x)$ (E) NOTA

The Laplace Transform is extremely important in the study of differential equations. The Laplace transform of a function $f(t)$ is defined to be $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$.

25) Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Find $\mathcal{L}\{e^{at}f(t)\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$.

(A) $F(s-a)$ (B) $F(a-s)$

(C) $\frac{F(s-a)}{s-a}$ (D) $\frac{F(a-s)}{a-s}$ (E) NOTA

26) Let $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$. Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Find $\mathcal{L}\{u(t-a)f(t-a)\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$.

(A) $e^{-a}\ F(s)$ (B) $e^{as}F(s)$

(C) $e^{-as}F(s+a)$ (D) $e^{as}F(s+a)$ (E) NOTA

27) Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Further, let $f(0) = f_0$ and $f'(0) = f_1$. Find $\mathcal{L}\{f''(t)\}$ in terms of these constants and $\mathcal{L}\{f(t)\} = F(s)$.

(A) $s^2F(s) - sf_0 + f_1$ (B) $s^2F(s) + sf_0 + f_1$

(C) $s^2F(s) - sf_0 - f_1$ (D) $-s^2F(s) + sf_0 + f_1$ (E) NOTA

28) Evaluate: $\mathcal{L}\{t^n\}$ for any non-negative integer n and any $s > 0$.

(A) $\frac{(n-1)!}{s^n}$ (B) $\frac{(n-1)!}{s^{n+1}}$

(C) $\frac{n!}{s^n}$ (D) $\frac{n!}{s^{n+1}}$ (E) NOTA

29) Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Find $\mathcal{L}\left\{\int_0^t f(x)dx\right\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$.

(A) $F\big(F(s)\big)$ (B) $\frac{F(s)}{s}$

(C) $sF(s)$ (D) $\frac{F(s)}{s^2}$ (E) NOTA

30) Find a second order linear differential equation with constant coefficients for which the following are solutions: $y_1(t) = e^{-2t}$, $y_2(t) = e^t$

A) $y'' - y' - 2y = 0$ B) $y'' + y' - 2y = 0$

C) $y'' + 2y' - 3y = 0$ D) $y'' - y' + 2y = 0$ E) NOTA