For all questions, answer choice (E) NOTA means that none of the given answers is correct. i is defined as $\sqrt{-1}$. DNE stands for Does Not Exist. Good luck and have fun!

1. Given the equation $3x^2 + 2x^5y^{-3} - 5y^{-2} = 4x^{-3}\ln(y)$, find $\frac{dx}{dy}$ evaluated at $x = 1$, $y = 1$.

A. 0 B. $1/3$ C. 1 D. 3 E. NOTA

2. Let $f(x) = -\frac{5}{3}x^3 + 3x + \frac{1}{2}x^4 + 10 - 2x^2$. Let (c, d) be the largest open interval on which the graph of f is both increasing and concave down. What is $c - 4d$?

A. -10 B. $-15/2$ C. $-7/3$ D. $1/3$ E. NOTA

3. If $g(x) = e^{x^3} - 3\ln x^2$, find $\lim_{h\to 0} \frac{g(x+2h)-g(x-2h)}{12h}$.

A. $\frac{1}{2}x^2e^{x^3} - \frac{1}{x}$ B. $x^2e^{x^3} - \frac{2}{x}$ C. $3x^2e^{x^3} - \frac{6}{x}$ D. $9x^2e^{x^3} - \frac{18}{x}$ E. NOTA

4. Evaluate the following limit: $\lim_{x\to 0^+} x\ln x$

A. 0 B. 1 C. e D. DNE E. NOTA

5. Given that $\sqrt[4]{2401} = 7$, approximate $\sqrt[4]{2417} - 7$ using differentials.

A. $\frac{1}{343}$ B. $\frac{4}{343}$ C. $\frac{16}{343}$ D. $\frac{64}{343}$ E. NOTA

6. Let f be a differentiable function such that $f(0) = 0, f(1) = 1, |f'(x)| \leq 3$ for all real numbers x. If the set of possible values of $\int_0^1 f(x)dx$ is the open interval (a, b), for real numbers a, b, determine $b - a$.

A. 1 B. $5/4$ C. $4/3$ D. 2 E. NOTA

7. Allais and Ellsberg are examining a sphere with radius 3. Allais wants to inscribe a cone with the largest possible volume inside the sphere, and Ellsberg wants to inscribe a cylinder with the largest possible volume inside the sphere. What is the square of the ratio of the radius of Allais's cone to the radius of Ellsberg's cylinder?

A. $2/3$ B. $3/4$ C. $4/3$ D. $3/2$ E. NOTA

8. Ed Glaeser has a giant inverted cone (missing the base) that he would like to fill with Diet Coke. The cone has a height that is three times its radius, and he is filling the cone at a rate of 4π cubic inches per second. How fast is the surface area of the base of the cone changing when the height of the Diet Coke is 12 inches?

A. $\frac{\pi}{3}$ B. $\frac{2\pi}{3}$ C. π D. 2π E. NOTA

9. For $x = 4 + e^t$ and $y = 5t^2$, find $\frac{d^2y}{dx^2}$.

A. $\frac{10e^t(1-t)}{e^{2t}}$ B. $\frac{10e^t(1-t)}{e^{3t}}$ C. $\frac{10(4+e^t-te^t)}{e^t(4+e^t)^2}$ D. $\frac{10t}{e^t}$ E. NOTA

10. If $\lim_{x\to\infty} \sqrt{Ax^2 + 10x} - Bx = 5$, what is $A + B$, for positive A, B?

A. 2 B. 4 C. 6 D. 8 E. NOTA

11. Evaluate the limit: $\lim_{x\to-\infty} 3x^2e^{2x}$

A. $-\frac{3}{2}$ B. $\frac{3}{2}$ C. 6 D. DNE E. NOTA

12. Evaluate the limit: $\lim_{x\to\infty} (2x^3 + e^{-x})^{\frac{1}{x^3}}$

A. 1 B. 2 C. e^2 D. $2e^2$ E. NOTA

13. For $f(x) = \arctan\left(\sec\left(\arcsin\frac{3}{\sqrt{9+x^2}}\right)\right)$, find $f'(3)$.

A. $-\frac{1}{6\sqrt{2}}$ B. $-\frac{1}{9\sqrt{2}}$ C. $\frac{1}{9\sqrt{2}}$ D. $1/3$ E. NOTA

14. Given $f(x) = 5x + 3e^x$, with inverse function $g(x)$, find $g'(3)$.

A. $\frac{1}{8}$ B. $\frac{1}{5}$ C. 5 D. 8 E. NOTA

15. Recall the epsilon-delta definition of a limit: we say that $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all x in the domain, $0 < |x - c| < \delta$ forces $|f(x) - L| < \epsilon$. For the limit $\lim_{x\to 1}(3x + 1) = 4$ and $\epsilon = 0.0015$, find the largest δ that works for the given ϵ.

A. 0.0001 B. 0.0005 C. 0.0015 D. 0.0045 E. NOTA

16. How many of the following limits are equal to zero?

I. $\lim_{x\to\infty} \frac{\sin x}{x}$

II. $\lim_{x\to\infty} \frac{\arctan x}{x}$

III. $\lim_{n\to\infty} \frac{n!}{e^n}$

IV. $\lim_{n\to\infty} \left(e + \frac{e}{n}\right)^{-n}$

V. $\lim_{x\to 0} \sqrt{x}$

A. 1 B. 2 C. 3 D. 4 E. NOTA

17. Given $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 3$, what is $\frac{d^2x}{dy^2}$ evaluated at $(8, 1)$?

A. 6 B. 4 C. $\frac{1}{4}$ D. $\frac{1}{6}$ E. NOTA

18. Evaluate the following limit: $\lim_{x\to 0^+}\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}$

A. 0 B. 1 C. $\sqrt{2}$ D. DNE E. NOTA

19. Find the sum of the x-coordinates of the local maxima and inflection points for $f(x) = \frac{1}{2}x^4 - \frac{5}{3}x^3 - 2x^2 + 12x + 5$.

A. $\frac{1}{6}$ B. $\frac{5}{3}$ C. $\frac{13}{6}$ D. $\frac{11}{3}$ E. NOTA

20. A 25-foot long ladder is propped up against a wall and is sliding down the wall at a rate of five feet per second. When the ladder is 15 feet up the wall, how fast is the bottom of the ladder sliding away from the wall?

A. $5/3$ B. $15/4$ C. 5 D. $20/3$ E. NOTA

21. Edouard is standing five feet from a wall with a rectangular screen on it. The bottom of the rectangular screen is twenty feet above the ground, and the top of the screen is fifteen feet higher. Edouard starts at one foot tall and grows at a rate of one foot every six seconds. When the angle of elevation from the top of Edouard's head to the bottom of the screen is shrinking at $\frac{1}{5}$ radians per minute, how fast is the angle of elevation from the top of Edouard's head to the top of the screen growing, in radians per minute?

A. $-\frac{1}{3}$ B. $-\frac{2}{37}$ C. $\frac{2}{37}$ D. $\frac{1}{3}$ E. NOTA

22. Evaluate the following: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2-i^2}}$

A. 0 B. 1 C. $\frac{\pi}{2}$ D. π E. NOTA

23. Evaluate the following at $x = 1$: $\frac{d}{dx}\int_{2x}^{3x^3} t \ln t \, dt$. If the answer to the evaluation is written in the form $a \ln b - c \ln d$, where b, d are prime, what is $a + c$?

A. 5 B. 13 C. 31 D. 35 E. NOTA

24. Evaluate the limit: $\lim_{x \to \infty} \frac{\sin 2x + 3x^2 + e^x}{-\cos x + 5x^5 + 3e^x}$

A. $\frac{1}{3}$ B. -1 C. 0 D. $\frac{3}{5}$ E. NOTA

25. Assume that one breath of air contains 10^{20} molecules, and that there are 10^{40} molecules of air in the entire atmosphere. Julius Caesar breathed his last breath on March 15, 44 BC. Assume for simplicity that all breaths contain exactly the same number of molecules, all air molecules are identical, the molecules in the atmosphere have not changed since Caesar's last breath, and that breathing one breath is sampling independently with replacement. You take a breath. Which of the following is closest to the probability that at least one molecule in this breath was also in Caesar's last breath?

A. 0 B. $\frac{1}{e}$ C. $1 - \frac{1}{e}$ D. 1 E. NOTA

26. For $f(x) = x^3 - x^2 + 3$, and using initial guess $x_0 = 1$, use Newton's method with two iterations to approximate a root of this function.

A. $-23/16$ B. $2/3$ C. $604/291$ D. $211/40$ E. NOTA

27. For $r = \sin\frac{\theta}{2} + 2\theta$, evaluate $\frac{dy}{dx}$ at $\theta = \pi$.

A. $2\pi + 1$ B. 2 C. $\pi + 1$ D. $\pi + \frac{1}{2}$ E. NOTA

Introducing partial derivatives! Let $f(x, y, z)$ be a function of three variables. The formal definition for the partial derivative of f with respect to x, denoted $\frac{\partial f}{\partial x}$, is $\lim_{h\to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h}$. Informally, you can just think of it as taking the derivative with respect to x, treating the other variables as constants. For instance, the partial derivative of $f(x, y) = x^2y$ with respect to x is $2xy$, and the partial derivative with respect to y is x^2.

28. Let's start off simple with partials! Let $f(x, y, z) = 5x^2yz - y^3z^2$. Evaluate $\frac{\partial f}{\partial y}$ at $(3, 2, 2)$.

A. -10 B. -7 C. 33 D. 42 E. NOTA

29. Just as there are higher-order single-variable derivatives, there are higher-order partial derivatives! The partial derivative $\frac{\partial^2 f}{\partial y \partial x}$ means that you first take the partial derivative with respect to x, then with respect to y. With that in mind, and using the function from the previous problem, calculate $\frac{\partial^3 f}{\partial y \partial x \partial z}$.

A. $10x$ B. $10xy$ C. $12x$ D. $12xz$ E. NOTA

30. Now for an application (in economics)! A firm has profit function

$$\pi(p, w) = pf(L) - wL$$

where p represents price of the good sold, w represents wage, L represents labor, and $f(L) = L^{0.5}$ represents the production function. Assume that p, w are fixed. What is the optimal level of labor to maximize profit (in terms of p, w)? To solve for this, note that the partial derivative of profit with respect to labor must be equal to zero.

A. $\frac{p^2}{4w^2}$ B. $\frac{p^2}{w^2}$ C. $\frac{w}{p}$ D. $\frac{p}{w}$ E. NOTA