Answers:

1. D
2. D
3. B
4. D
5. E
6. D
7. D
8. B
9. A
10. C
11. B
12. C
13. A
14. C
15. D
16. B
17. B
18. B
19. A
20. E
21. B
22. B
23. A
24. C
25. B
26. E
27. A
28. D
29. A
30. D

Solutions:

1. It is well known that $\frac{a}{1-r} = a + ar + ar^2 + \ldots$ for all $-\infty < a < \infty, -1 < r < 1$. if r is outside of this range, then either $r = 1$, in which case $\frac{a}{1-r}$ is undefined and $a + ar + ar^2 + \ldots$ diverges, or $r \neq 1$, in which case $\frac{a}{1-r}$ is some real number, and the righthand side diverges.
   Thus, in our case, $r = 9x$, so $-1 < 9x < 1$. Dividing by 9 yields $\frac{-1}{9} < x < \frac{1}{9}$.
   D
2. Let $a_k = \frac{(-1)^k x^2}{e^k}$. Perform the ratio test on this sequence: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{(-1)^{k+1} x^2}{e^{k+1}} \cdot \frac{e^k}{(-1)^k \cdot x^2} = -e^{-1} = r$. Clearly, $-1 < r < 1$. Thus, this series converges for any x.
   Note: The series $\sum_{k=0}^{\infty} \frac{(-1)^k}{e^k}$ converges, so multiplying it by x^2 will multiply the sum by a constant without changing convergence.
   D
3. $\sqrt[3]{12\sqrt[3]{12\sqrt[3]{12\ldots}}} = 12^{1/3} \cdot (12^{1/3})^{1/3} \cdot ((12^{1/3})^{1/3})^{1/3} \ldots = 12^{\frac{1}{3}+(\frac{1}{3})^2+(\frac{1}{3})^3+\ldots} = 12^{\frac{1/3}{1-(1/3)}} = 12^{\frac{1/3}{2/3}} = 12^{1/2} = 2\sqrt{3}$
   B
4. $\sqrt{12 + \sqrt{12 + \sqrt{12 + \cdots}}} = x = \sqrt{12 + x} \rightarrow x^2 = 12 + x \rightarrow 0 = x^2 - x - 12 = (x-4)(x+3)$

   Because x is a square root of a sum of positive numbers, $x > 0 \rightarrow x = 4$.
   D

5. Let us consider each of I, II, and III separately.

   I:
   Consider the sequence $a_n = (\frac{1}{1}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots)$. It is well known that $\sum_{k=1}^{\infty} a_k = ln(2)$. However, consider the sequence b_n for which $b_i = a_{2i}$. Since every element in sequence b_n is also in a_n, b_n is a subsequence of a_n. However, $\sum_{k=1}^{\infty} b_k = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \ldots = -\frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\right)$. Since the latter series diverges, $\sum_{k=1}^{\infty} b_k$ also diverges. However, $\sum_{k=1}^{\infty} a_k$ converges. Therefore, (I) need not be true.

   II:
   Consider the sequence $a_n = \frac{(-1)^n}{\sqrt{n}}$. Clearly, $|a_n|$ is decreasing and $\lim_{i\to\infty} a_i = 0$. Thus, by

the Alternating Series Test, $\sum_{k=1}^{\infty} a_k$ converges. However, observe that $a_n{}^2 = \frac{(-1)^{2n}}{n} = \frac{1}{n}$. It is well known that $\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} a_k{}^2$ diverges. Therefore, (II) need not be true.

III:

Consider the sequence $a_n = \frac{(-1)^n}{n}$. Clearly, $|a_n|$ is decreasing, and $\lim_{i\to\infty} a_i = 0$. Thus, by the Alternating Series Test, $\sum_{k=1}^{\infty} a_k$ converges. However, observe that $|a_n| = \frac{|(-1)^n|}{|n|} = \frac{1}{n}$. It is well known that $\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} |a_k|$ diverges. Therefore, (III) need not be true.

Since none of (I, II, III) must be true, the answer is: None

E.

6. The given limit is of the form of a Reimann Sum. In a Reimann Sum representing $\int_a^b f(x)dx$, the $\frac{i}{n}$ term represents x, the operations performed on the $\frac{i}{n}$ term represent $f(x)$, the $\frac{1}{n}$ term represents dx, and the bounds of the sum represent the limits of integration (a finite number represents 0, n represents 1, $2n$ represents 2, etc.). Using this information, $f(x) = (1+x)^3$, and the limits of integration are 0 and 3. $\int_0^3 (1+x)^3\, dx = [\frac{(1+x)^4}{4}]_0^3 = \frac{4^4}{4} - \frac{1^4}{4} = \frac{256-1}{4} = \frac{255}{4}$.

   D

7. $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$ implies that $3^x = e^{ln(3)\cdot x} = \sum_{k=0}^{\infty} \frac{(ln(3)\cdot x)^k}{k!}$.

   D.

8. Call the given sum $S = \sum_{n=1}^{\infty} \frac{n^2}{2^{2n}} = \frac{1}{4} + \frac{4}{16} + \frac{9}{64} + \frac{16}{256} + \ldots$. Dividing by 4 gives $\frac{S}{4} = \frac{1}{16} + \frac{4}{64} + \frac{9}{256} + \ldots$. Subtracting the two previous equations yields $\frac{3S}{4} = \frac{1}{4} + \frac{3}{16} + \frac{5}{64} + \frac{7}{256} + \ldots$. Dividing by 4 gives $\frac{3S}{16} = \frac{1}{16} + \frac{3}{64} + \frac{5}{256} + \ldots$. Subtracting the two previous equations yields $(\frac{12S}{16} - \frac{3S}{16}) = \frac{9S}{16} = \frac{1}{4} + \frac{2}{16} + \frac{2}{64} + \frac{2}{256} + \ldots = -\frac{1}{4} + (\frac{2}{4} + \frac{2}{16} + \frac{2}{64} + \ldots) = -\frac{1}{4} + \frac{2/4}{1-\frac{1}{4}} = -\frac{1}{4} + \frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{1}{4} + \frac{2}{3}$ $= \frac{-3+8}{12} = \frac{5}{12} = \frac{9S}{16} \Rightarrow S = \frac{5}{12} \cdot \frac{16}{9} = \frac{20}{27}$.

   B

9. First, notice that $\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n\cdot 4^n}$ is completely symmetric about $x = 4$ because the only term involving $x - 4$ is raised to $2n$, an even power. Therefore, a number r is within the interval of convergence if and only if $4 - (r - 4) = 8 - r$ is also within the interval of convergence.

   In order to determine the interval of convergence, let us use the ratio test. Calling the sequence a_n, we have $\lim_{n\to\infty} \frac{a_{n+!}}{a_n} = \lim_{n\to\infty} \frac{(x-4)^{2(n+1)}}{(n+1)\cdot 4^{n+1}} / \frac{(x-4)^{2n}}{n\cdot 4^n} = \lim_{n\to\infty} \frac{(x-4)^{2(n+1)}\cdot n\cdot 4^n}{(n+1)\cdot 4^{n+1}\cdot (x-4)^{2n}} =$ $\lim_{n\to\infty} \frac{(x-4)^2}{4}$. Thus, if $\lim_{n\to\infty} \frac{(x-4)^2}{4} < 1$, the series converges, and if $\lim_{n\to\infty} \frac{(x-4)^2}{4} > 1$, the series diverges. From this fact and from the symmetry about $x = 4$, the interval of convergence must be either $(2,6)$ or $[2,6]$, depending on whether the series converges at $x = 6{,}2$. Plugging in $x = 6$, the given series is $\sum_{n=1}^{\infty} \frac{2^{2n}}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Thus, the interval of convergence is $(2,6)$

A.

10. The motivation behind this solution is the intuition that the graph of $f(x) = e^{-x}|sin(x)|$ has periodic "humps" with a period of π, the same period as $|sin(x)|$. The area of these humps can be calculated using a geometric series.

    More rigorously:
    $|sin(x+\pi)| = |sin(x)cos(\pi) + sin(\pi)cos(x)| = |-sin(x)| = |sin(x)|$.
    Call $f(x) = e^{-x}|sin(x)|$. $f(x+\pi) = e^{-x-\pi}|sin(x+\pi)| = e^{-\pi} \cdot e^{-x}|sin(x)| = e^{-\pi} f(x)$.
    The question asks to evaluate $\int_0^{\infty} (e^{-x}|sin(x)|)\, dx$. We can "split" this integral into intervals of width π by rewriting the integral as $\sum_{n=0}^{\infty} [\int_{n\pi}^{(n+1)\pi} (e^{-x}|sin(x)|)\, dx]$.
    Additionally, using previously obtained information, $\int_{n\pi}^{(n+1)\pi} f(x)\, dx = \int_{(n-1)\pi}^{n\pi} f(x+\pi)\, dx = \int_{(n-1)\pi}^{n\pi} e^{-\pi} f(x)\, dx = e^{-\pi} \cdot \int_{(n-1)\pi}^{n\pi} f(x)\, dx$. Thus, the area of each interval is $e^{-\pi}$ times the area of the previous integral. Therefore, $\sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} (e^{-x}|sin(x)|)\, dx = \frac{\int_0^{\pi} e^{-x}|sin(x)|)\, dx}{1-e^{-x}}$.
    Evaluate $\int_0^{\pi} (e^{-x}|sin(x)|)\, dx = \int_0^{\pi} e^{-x} \cdot sin(x)\, dx$ by using integration by parts:

    $$\int_0^{\pi} e^{-x} \cdot sin(x)\, dx = [-e^{-x} sin(x)]_0^{\pi} - \int_0^{\pi} -e^{-x} \cdot cos(x)\, dx$$

    $$= (0-0) + \{[-e^{-x} cos(x)]_0^{\pi} - \int_0^{\pi} -e^{-x} \cdot -sin(x)\, dx\}$$

    $$= (-e^{-\pi} \cdot (-1) - (-1) \cdot 1) - \int_0^{\pi} e^{-x} \cdot sin(x)\, dx \Rightarrow 2\int_0^{\pi} e^{-x} \cdot sin(x)\, dx = e^{-\pi} + 1$$

    $$\Rightarrow \int_0^{\pi} e^{-x} \cdot sin(x)\, dx = \frac{e^{-\pi} + 1}{2}$$

    Therefore, $\int_0^{\infty} (e^{-x}|sin(x)|)\, dx = \frac{\frac{e^{-\pi}+1}{2}}{1-e^{-\pi}} = \frac{e^{-\pi}+1}{2(1-e^{-\pi})}\left(\cdot \frac{e^{\pi}}{e^{\pi}}\right) = \frac{e^{\pi}+1}{2(e^{\pi}-1)}$

    C.

11. Ben: $\int_0^6 e^x dx = [e^x]_0^6 = e^6 - 1 = B$.

    Zhao: Each interval has width $\frac{6}{2} = 3$. Thus, $\frac{e^0+e^3}{2} \cdot 3 + \frac{e^3+e^6}{2} \cdot 3 = \frac{3}{2} + 3e^3 + \frac{3}{2}e^6 = Z$.

    $Z - B = \frac{3}{2} + 3e^3 + \frac{3}{2}e^6 - (e^6 - 1) = \frac{5}{2} + 3e^3 + \frac{1}{2}e^6$.

    B.

12. Generally, the Taylor expansion about $x = a$ for $f(x) = e^x$ is $\sum_{n=0}^{\infty} e^a \frac{(x-a)^n}{n!}$. Thus, the 2nd degree approximations that each person uses are as follows:

    Jonathan: $\frac{1}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} = 1 + x + x^2$

    Henrik: $e^6\left(\frac{1}{0!} + \frac{(x-6)^1}{1!} + \frac{(x-6)^2}{2!}\right)$

Defining J as the integral that Jonathan calculates and H as the integral Henrik calculates:

$J = \int_0^6 (1 + x + x^2)dx = [x + \frac{x^2}{2} + \frac{x^3}{6}]_0^6 = 6 + 18 + 36 = 60.$

$H = \int_0^6 e^6(1 + \frac{x-6}{1} + \frac{(x-6)^2}{2})dx = e^6[x + \frac{(x-6)^2}{2} + \frac{(x-6)^3}{6}]_0^6 = e^6(6 + 0 + 0 - (0 + 18 - 36)) = 24e^6.$

To answer whose approximation is more accurate, we compare each approximation to the actual integral, which we determined in Question 10 to be $e^6 - 1 = I$.

$|J - I| = |e^6 - 1 - 60| = |e^6 - 61|$. $3 > e = 2.7\ldots > 2 \Rightarrow 3^6 = 729 > e^6 > 64 = 2^6$. Therefore, $668 > |e^6 - 61| > 3$.

$|H - I| = |e^6 - 1 - 24e^6| = |23e^6 + 1|$. $e^6 > 64$. Therefore, $|23e^6 + 1| > 1473 = 23 \cdot 64 + 1$. Therefore, Jonathan's integral approximation is more accurate.

Additionally, the difference between the two approximations is $24e^6 - 60$

C.

13. Factoring the bottom polynomial gives $\frac{n-1}{n^3+10n^2+19n-30} = \frac{n-1}{(n+5)(n-1)(n+6)} = \frac{1}{(n+5)(n+6)}$.

Now, using partial fraction decomposition, $\frac{1}{(n+5)(n+6)} = \frac{A}{(n+5)} + \frac{B}{(n+6)}$. $A(n+6) + B(n+5) = 1 \Rightarrow A + B = 0, 6A + 5B = 1 \Rightarrow A = 1, B = -1$. Thus, the given series is $\sum_{n=4}^{\infty}(\frac{1}{n+5} - \frac{1}{n+6}) = (\frac{1}{9} - \frac{1}{10}) + (\frac{1}{10} - \frac{1}{11}) + (\frac{1}{11} - \frac{1}{12}) + \ldots = \frac{1}{9} + (-\frac{1}{10} + \frac{1}{10}) + (-\frac{1}{11} + \frac{1}{11}) + \ldots = \frac{1}{9}$

A.

14. Let $x = 2 + \frac{3}{2+\frac{3}{2+\ldots}}$, $x = 2 + \frac{3}{x} \to x - 2 - \frac{3}{x} = 0 = x^2 - 2x - 3 = (x-3)(x+1)$. Because the given continued fraction contains only positive numbers being summed or divided, $x > 0 \to x = 3$

C.

15. Write the sum as $\sum_{n=0}^{\infty}\frac{(n+2)^2}{n!} = \sum_{n=0}^{\infty}\frac{n^2}{n!} + \frac{4n}{n!} + \frac{4}{n!} = \sum_{n=0}^{\infty}\frac{n^2}{n!} + \sum_{n=0}^{\infty}\frac{4n}{n!} + \sum_{n=0}^{\infty}\frac{4}{n!}$.

$\sum_{n=0}^{\infty}\frac{4}{n!} = 4\sum_{n=0}^{\infty}\frac{1}{n!} = 4e.$

$\sum_{n=0}^{\infty}\frac{4n}{n!} = 0 + \sum_{n=1}^{\infty}\frac{4}{(n-1)!} = 4\sum_{n=1}^{\infty}\frac{1}{(n-1)!} = 4\sum_{n=0}^{\infty}\frac{1}{n!} = 4e.$

$$\sum_{n=0}^{\infty}\frac{n^2}{n!} = 0 + \sum_{n=1}^{\infty}\frac{n}{(n-1)!} = \sum_{n=1}^{\infty}\frac{n-1}{(n-1)!} + \sum_{n=1}^{\infty}\frac{1}{(n-1)!} = (0 + \sum_{n=2}^{\infty}\frac{n-1}{(n-1)!}) + \sum_{n=0}^{\infty}\frac{1}{n!}$$

$$= (\sum_{n=2}^{\infty}\frac{1}{(n-2)!}) + e = (\sum_{n=0}^{\infty}\frac{1}{n!}) + e = e + e = 2e$$

Therefore, $\sum_{n=0}^{\infty}\frac{(n+2)^2}{n!} = 4e + 4e + 2e = 10e$

D.

16. Evaluate each case:

I:

Use the integral test for this series. The corresponding integral to this series is

$\int_2^\infty \frac{1}{x \cdot ln(x)} dx$. Perform a u substitution, with $u = ln(x),\ du = \frac{dx}{x}$, then $\int_2^\infty \frac{1}{x \cdot ln(x)} dx = \int_{ln(2)}^\infty \frac{du}{u} = [ln(u)]_{ln(2)}^\infty$. At $u = \infty,\ ln(u)$ is infinite, so the integral diverges. The Integral Test Implies that the given series also diverges.

II:

Notice that for positive $n, n^n > n! = n \cdot (n-1) \cdot \ldots \cdot 1$. Therefore, for positive n, $ln(n^n) = n \cdot ln(n) > ln(n!)$, and thus $\frac{1}{n \cdot ln(n)} < \frac{1}{ln(n!)}$. Therefore, $\sum_{n=2}^\infty \frac{1}{n \cdot ln(n)} < \sum_{n=2}^\infty \frac{1}{ln(n!)}$. From part I of this question, we know that $\sum_{n=2}^\infty \frac{1}{n \cdot ln(n)}$ diverges. By the Direct Comparison Test, $\sum_{n=2}^\infty \frac{1}{ln(n!)}$ also diverges.

III:

$\sum_{n\,=\,2}^\infty \frac{1}{ln(n)^n} = \sum_{n\,=\,2}^\infty \frac{1}{e^{ln(ln(n)^n)}} = \sum_{n\,=\,2}^\infty \frac{1}{e^{n \cdot ln(ln(n))}}$.

Define $a_n = \frac{1}{e^{n \cdot ln(ln(n))}}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{e^{(n+1)\cdot ln(ln(n+1))}}}{\frac{1}{e^{n \cdot ln(ln(n))}}}\right| = \lim_{n\to\infty} \left|\frac{e^{n \cdot ln(ln(n))}}{e^{(n+1)\cdot ln(ln(n+1))}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{e^{n \cdot ln(ln(n))}}{e^{n \cdot ln(ln(n+1)}} \cdot \frac{1}{e^{ln(ln(n+1))}}\right| = \lim_{n\to\infty} \left|(\frac{e^{ln(ln(n))}}{e^{ln(ln(n+1)}})^n \cdot \frac{1}{ln(n+1)}\right|$$

$= \lim_{n\to\infty} \left|(\frac{ln(n)}{ln(n+1)})^n \cdot \frac{1}{ln(n+1)}\right|$. $ln(n) < ln(n+1)$. $\lim_{n\to\infty} \frac{1}{ln(n+1)} = 0$. Therefore, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, and by the Ratio Test, $\sum_{n\,=\,2}^\infty \frac{1}{ln(n)^n}$ converges.

B.

17. Write $f(x)$ as $g(x) \cdot h(x)$, where $g(x) = e^x, h(x) = \frac{x}{1-x^2}$. Consider $\frac{x}{1-x^2}$ to be an infinite geometric series with $a = x\ \ and\ \ r = x^2$. Thus, $\frac{x}{1-x^2} = (x)(1 + x^2 + x^4 + \ldots)$. Additionally, $e^x = 1 + x + \frac{x^2}{2!} + \ldots$. Therefore, $f(x) = g(x) \cdot h(x) = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots)(x)(1 + x^2 + x^4 + \ldots) = x(1 + x + (1 + \frac{1}{2!})x^2 + (1 + \frac{1}{3!})x^3 + (1 + \frac{1}{2!} + \frac{1}{4!})x^4 + \ldots)$. The 4th degree Maclaurin Series Approx. is therefore $M_4(x) = x + x^2 + (1 + \frac{1}{2!})x^3 + (1 + \frac{1}{3!})x^4 = x + x^2 + \frac{3}{2}x^3 + \frac{7}{6}x^4$. $M_4(1) = 1 + 1^2 + \frac{3}{2}1^3 + \frac{7}{6}1^4 = \frac{6+6+9+7}{6} = \frac{28}{6} = \frac{14}{3}$

B.

18. This series is of the form $\sum_{n=1}^\infty \frac{x^n}{n}$, for which $x = -\frac{1}{2}$. If we consider this series to be a function $f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ evaluated at $x = -\frac{1}{2}$, we can differentiate each term to obtain $f'(x) = 1 + x + x^2 + \ldots + x^i + \ldots$. For $x \in (-1, 1)$, $f'(x)$ is a geometric series with first term 1 and common ratio x, which implies $f'(x) = \frac{1}{1-x} = \frac{df}{dx}$. Thus, $f = \int \ \frac{dx}{1-x} =$

$-ln|1 - x| + C$. To determine C, evaluate $f(0) = 0 + \frac{0^2}{2} + \frac{0^3}{3} + \ldots = 0 = -ln|1 - 0| + C = 0 + C = C = 0$. Therefore, the answer is $f(-\frac{1}{2}) = -ln|1 + \frac{1}{2}| = -ln(\frac{3}{2}) = ln(\frac{2}{3})$.

B.

19. Notice that the denominator may be factored as $x^2 + 7x + 12 = (x + 3)(x + 4)$. Use partial fraction decomposition to obtain $\frac{1}{(x+3)(x+4)} = \frac{1}{x+3} - \frac{1}{x+4}$. Thus, our sum is $\sum_{x=0}^{20}[\frac{1}{x+3} - \frac{1}{x+4}] = (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) + \ldots + (\frac{1}{22} - \frac{1}{23}) + (\frac{1}{23} - \frac{1}{24})$

$$= \frac{1}{3} + (-\frac{1}{4} + \frac{1}{4}) + (-\frac{1}{5} + \frac{1}{5}) + \ldots + (-\frac{1}{23} + \frac{1}{23}) + (-\frac{1}{24}) = \frac{1}{3} - \frac{1}{24}$$

$$= \frac{8 - 1}{24} = \frac{7}{24}.$$

A

20. Rewrite a_i as $a_i = \frac{(x+1)^{1/i} - x^{1/i}}{1}$. Now, rationalize the numerator of a_i, as follows: $a_i = \frac{(x+1)^{1/i} - x^{1/i}}{1} \cdot \frac{(x+1)^{(i-1)/i}x^0 + (x+1)^{(i-2)/i}x^{1/i} + (x+1)^{(i-3)/i}x^{3/i} + \ldots + (x+1)^{1/i}x^{(i-2)/i} + (x+1)^0 x^{(i-1)/i}}{(x+1)^{(i-1)/i}x^0 + (x+1)^{(i-2)/i}x^{1/i} + (x+1)^{(i-3)/i}x^{3/i} + \ldots + (x+1)^{1/i}x^{(i-2)/i} + (x+1)^0 x^{(i-1)/i}}$

$$= \frac{(x+1) - x}{(x+1)^{(i-1)/i}x^0 + (x+1)^{(i-2)/i}x^{1/i} + (x+1)^{(i-3)/i}x^{3/i} + \ldots + (x+1)^{1/i}x^{(i-2)/i} + (x+1)^0 x^{(i-1)/i}}$$

$$= \frac{1}{(x+1)^{(i-1)/i}x^0 + (x+1)^{(i-2)/i}x^{1/i} + (x+1)^{(i-3)/i}x^{3/i} + \ldots + (x+1)^{1/i}x^{(i-2)/i} + (x+1)^0 x^{(i-1)/i}}$$

In the Question, we are told to evaluate $\sum_{k=1}^{\infty} a_k(1)$. Let us write $a_1(1)$ as $a_i(1) = \frac{1}{(1+1)^{(i-1)/i}1^0 + (1+1)^{(i-2)/i}1^{1/i} + (1+1)^{(i-3)/i}1^{3/i} + \ldots + (1+1)^{1/i}1^{(i-2)/i} + (1+1)^0 1^{(i-1)/i}}$

$= \frac{1}{2^{(i-1)/i} + 2^{(i-2)/i} + 2^{(i-3)/i} + \ldots + 2^{1/i} + 2^0}$. Written in this form, we can see that the denominator has i terms being summed, all of which are positive, and less than $2^1 = 2$. This implies that $\frac{1}{a_i(1)} < 2 \cdot i$, for all positive i. Therefore, $a_i(1) > \frac{1}{2i}$.

Additionally, $\sum_{i=1}^{\infty} \frac{1}{2i} = \frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{i}$, which diverges (Harmonic Series). Therefore, $\sum_{k=1}^{\infty} a_k(1)$ diverges.

E

21. A_n, the area of one petal, is the area under the polar curve from two consecutive instances of $sin(n\theta) = 0$. The 2 lowest positive values for which $sin(n\theta) = 0$ are $n\theta = 0, \pi$. Thus, $A_n = \frac{1}{2}\int_0^{\pi/n} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/n} sin^2(n\theta)\, d\theta$. Using u substitution, $u = n\theta,\ \ du = n\, d\theta,\ \ A_n = \frac{1}{2}\int_0^{\pi} sin^2(u)\ \frac{du}{n} = \frac{1}{2n}\int_0^{\pi} sin^2(u)\, du$. Therefore, $\lim_{n\to\infty}[n \cdot A_n] = \lim_{n\to\infty}[n \cdot \frac{1}{2n}\int_0^{\pi} sin^2(u)\, du] = \frac{1}{2}\int_0^{\pi} sin^2(u)\, du = \frac{1}{2}\int_0^{\pi} \frac{1}{2} - \frac{cos(2u)}{2}\, du = \frac{1}{2}\left[\frac{u}{2} - \left(\frac{sin(2u)}{2 \cdot 2}\right)\right]_0^{\pi}$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{0}{2} - \left(\frac{sin(2\pi)}{4} - \frac{sin(0)}{4}\right)\right] = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

B.

22. Define the nth triangular number, T_n, as $\frac{n(n+1)}{2}$. Thus, the series we must evaluate is $\sum_{n=1}^{\infty} \frac{1}{T_n} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 2 \cdot \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = 2 \cdot \frac{1}{1} = 2$

B

23. $\lim_{n\to\infty} \sum_{k=2n}^{8n} \frac{1}{k\cdot(\ln(k)-\ln(n))} = \lim_{n\to\infty} \sum_{k=2n}^{8n} \frac{1}{k\cdot\ln\left(\frac{k}{n}\right)} = \lim_{n\to\infty} \sum_{k=2n}^{8n} \frac{\frac{1}{n}}{\frac{k}{n}\cdot\ln\left(\frac{k}{n}\right)} =$ $\lim_{n\to\infty} \sum_{k=2n}^{8n} \frac{1}{n} \cdot \frac{1}{\frac{k}{n}\cdot ln(\frac{k}{n})} = S$. Interpreting S as a Riemann Sum representing an integral, $S = \int_2^8 \frac{1}{x\cdot ln(x)} dx$. $u = ln(x), du = \frac{dx}{x}$. $S = \int_{ln(2)}^{ln(8)} \frac{du}{u} = [ln(u)]_{ln(2)}^{ln(8)} = ln(ln(8)) - ln(ln(2)) = ln(\frac{ln(8)}{ln(2)})$

$$= ln(\frac{3 \cdot ln(2)}{ln(2)}) = ln(3)$$

A.

24. $f(x) = 2x + \frac{2}{3}x^3 + \frac{6}{5}x^5 + \frac{6}{7}x^7 + \frac{10}{9}x^9 + \frac{10}{11}x^{11} + \ldots = (1+1)x + (1-\frac{1}{3})x^3 + (1+\frac{1}{5})x^5 + (1-\frac{1}{7})x^7 + \ldots$

$= (x + x^3 + x^5 + x^7 + \ldots) + (x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots)$. The first term of this sum is an infinite geometric series with first term x and common ratio x^2, while the second term is the Taylor Series for $arctan(x)$. Therefore, $f(x) = \frac{x}{1-x^2} + arctan(x)$. $f(\frac{\sqrt{3}}{3}) = \frac{\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}^2} + arctan(\frac{\sqrt{3}}{3}) = \frac{\frac{\sqrt{3}}{3}}{1-\frac{1}{3}} + \frac{\pi}{6} = \frac{\sqrt{3}}{3} \cdot \frac{3}{2} + \frac{\pi}{6} = \frac{3\sqrt{3}+\pi}{6}$

C.

25. Multiply a_n by $\frac{e^{-n}}{e^{-n}} = 1$ to obtain $a_n = \frac{e^{-n}2^n + e^{-n}e^{n-1}}{e^{-n}n^{2019} + e^{-n}2^n + e^{-n}e^n} = \frac{(2/e)^n + e^{-1}}{e^{-n}n^{2019} + (\frac{2}{e})^n + 1}$.

Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2/e)^n + e^{-1}}{e^{-n}n^{2019} + (\frac{2}{e})^n + 1} = \frac{\lim_{n\to\infty}[(2/e)^n] + e^{-1}}{\lim_{n\to\infty}[e^{-n}n^{2019} + (2/e)^n] + 1} = \frac{0 + e^{-1}}{(0+0)+1} = e^{-1}$.

B.

26. First, the ball travels down by 5. Then, it bounces back up to 3, then down 3, then up $3 \cdot \frac{3}{5} = \frac{9}{5}$, then down $\frac{9}{5}$, and so on. Thus, The total distance travelled by the ball is $5 + 2(3) + 2(\frac{9}{5}) + 2(\frac{27}{25}) + \ldots = 5 + 2(3 + \frac{9}{5} + \frac{27}{5} + \ldots) = 5 + 2(\frac{3}{1-3/5})$

$= 5 + 2(\frac{3}{2/5}) = 5 + 2(\frac{15}{2}) = 5 + 15 = 20.$

E

27. Generally, the Taylor Series of $\sin(z)$ about 0 is: $sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. Plugging in $z = x$ and multiplying by x yields $x \cdot sin(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

A.

28. Generally, the Taylor Series of $sin(z)$ about 0 is: $sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. Plugging in $z = 2x^2$ and dividing by 2 yields $\frac{sin(2x^2)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n+1}}{2 \cdot (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n} \cdot x^{4n+2}}{(2n+1)!}$

D.

29. $\lim_{n\to\infty} \sum_{x=0}^{n} \sqrt{\frac{x^2}{n^4} - \frac{x^4}{n^6}} = \lim_{n\to\infty} \sum_{x=0}^{n} \frac{1}{n} \cdot \frac{x}{n} \cdot \sqrt{1 - \frac{x^2}{n^2}} = S$. Interpreting S as a Riemann Sum, $\frac{1}{n} = dy, \frac{x}{n} = y,\ S = \int_0^1 y \cdot \sqrt{1 - y^2} dy$. Perform a u substitution, $u = (1 - y^2), du = -2y\, dy. S = \int_1^0 \frac{\sqrt{u}}{-2} du = \int_0^1 \frac{\sqrt{u}}{2} du = [\frac{2}{3} \cdot \frac{u^{3/2}}{2}]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

A.

30. If $n = 0$, then $A_n = \int_0^{\pi} sin(nx) dx = \int_0^{\pi} sin(0) dx = \int_0^{\pi} 0\, dx = 0.$ Otherwise, $A_n = \int_0^{\pi} sin(nx)\, dx = [-\frac{cos(nx)}{n}]_0^{\pi} = -\frac{cos(n\pi)}{n} - (-\frac{1}{n}) = \frac{-cos(n\pi)+1}{n}$. If n is even, $cos(n\pi) = 1, A_n = \frac{-1+1}{n} = 0$. If n is odd, $cos(n\pi) = -1, A_n = \frac{-(-1)+1}{n} = \frac{2}{n}$. Thus, $\sum_{n=0}^{\infty} A_n = 0 + \frac{2}{1} + 0 + \frac{2}{3} + 0 + \frac{2}{5} + \ldots$. This is a Harmonic Series (the denominators are an arithmetic series), and therefore this sum diverges

D.