Theta Area & Volume Answers

1. A
2. C
3. D
4. E
5. C
6. D
7. A
8. E
9. C
10. B
11. B
12. C
13. E
14. B
15. A
16. C
17. C
18. D
19. B
20. C
21. D
22. B
23. A
24. C
25. B
26. E
27. B
28. E
29. B
30. A

Theta Area & Volume Solutions

(1) A An equilateral triangle with side length $\mathbf{4}$ has area $\mathbf{4\sqrt{3}}$.

(2) C A square with diagonal 4 has side length $2\sqrt{2}$. Squaring this we get 8.

(3) D An octagon with side length s has area $2s^2(1+\sqrt{2})$. Plugging in 4 we get $32(1+\sqrt{2})$.

(4) E A rhombus is not well defined without an angle in addition to side length. The area of a rhombus side length 4 can in fact take on any value from 0 to 16 depending on the angle between side lengths.

(5) C Half the major axis is $a = 5/2$ and half the minor axis is $b = 3/2$. Plug into the formula $A = ab\pi$ to get $15\pi/4$.

(6) D The diagonal of a cube is $s\sqrt{3}$. The side length of this cube is $4/\sqrt{3}$, so the volume is $64/3\sqrt{3}$, which is $64\sqrt{3}/9$.

(7) A An octahedron is composed of two square pyramids. The altitude of one of these pyramids is half the length of the space diagonal, so $h = 2$. The base of one of these pyramids is the area of a square with diagonal 4, so $b = 8$. Therefore, the volume of the octahedron is $32/3$.

(8) E Half the major axis is $a = 3$ and half the minor axis is $b = 2$. Since we are rotating about the minor axis, $c = 3$. Thus plugging into the formula $V = \frac{4}{3}abc\pi$ we get 24π.

(9) C The volume of Brooke's clothing is $\frac{3}{4}(400\pi)(40)(2) = 24000\pi\ cm^3$. The volume of the dryer is $(900\pi)(40) = 36000\pi\ cm^3$. Thus $p = 2/3$. Solving for t, we get $t = 90$ minutes.

(10) B The shape is a circle. Solving for radius r: $2\pi r = 2 \Longrightarrow r = 1/\pi \Longrightarrow \pi r^2 = 1/\pi$.

(11) B It is well known that the ratio between the width and length of the pigpen is $1:2$, thus the maximum area is 3200.

(12) C The hexagon has side length 6, so area $54\sqrt{3}$.

(13) E Be sure to sort the coordinates to have a convex quadrilateral. An order is $(1,2), (-1,3), (0,5), (6,1)$. This yields an area of $19/2$.

(14) B Dropping the altitude, we see that we split the side into segments length 1 and 4 with an altitude of 3. This yields two cylinders both with radius 3 one of height 1 and the other of height 4. Thus $\frac{9\pi(1+4)}{3} = 15\pi$.

(15) A Quadrupling the great circle of the sphere is the same as doubling the radius, which means $V_1 = 8V_0 \Longrightarrow \frac{V_1}{V_0} = 8$.

(16) C The area of the annulus is half the chord's length squared times π, so 16π.

(17) C He has a $\frac{1}{9}, \frac{1}{3}, \frac{5}{9}$ probability of hitting the $10, 4, 1$ point regions respectively. Taking a weighted average: $\frac{10}{9} + \frac{4}{3} + \frac{5}{9} = 3$.

(18) D At a given point, the radius of the circular path is constant at r. The height from the center to the final grade is 15, and the distance from Erin to the final grade is 17. Since these lengths form a right triangle, $r = 8$, so $A = \pi r^2 = 64\pi$.

(19) B Let X be the center of the circle. Note that CXA and $C'XA$ are isosceles triangles with base angle $30°$ with side lengths $2,2,2\sqrt{3}$. These have a combined area of $2\sqrt{3}$. The remaining portion of the area is the $120°$ sector bound by radii CX and $C'X$. This has area $\frac{4\pi}{3}$. Thus the area is $2\sqrt{3} + \frac{4\pi}{3}$.

(20) C It is well known that 4 lines can split a plane into a maximum of 11 regions of positive area.

(21) D The plane cuts off a tetrahedron with base area $1/2$ and height 1. The tetrahedron has volume $1/6$, so the remaining volume is $\frac{5}{6} > \frac{1}{6}$, thus $5/6$.

(22) B Extend the edges to form a right hexagonal pyramid. The ratio of the bases is $\frac{54}{96} = \frac{9}{16}$. So the ratio of the height of the frustum to the height of the pyramid is $\frac{1}{4}$. Thus the frustum is $1 - \frac{27}{64} = \frac{37}{64}$ of the volume of the entire pyramid. The pyramid has height 4 and base $96\sqrt{3}$, which means the volume of the frustum is $\frac{(4\times 96\sqrt{3})}{3} \times \frac{37}{64} = 74\sqrt{3}$.

(23) A A cyclic trapezoid must be isosceles, so this is half a regular hexagon with side length 5. The circumradius is 5, so the area is 25π.

(24) C Since the triangles share a height, by same height different base, the ratio of the areas of the triangles is the ratio of the bases. $\frac{X_{21}X_{84}}{X_2X_5} = \frac{63}{3} = 21$.

(25) B We use geometric probability (which is an application of areas!). For the economics group, we subtract out the isosceles right triangles from the whole 2 hour by 2 hour square.

$$P = 1 - \left(1 - \frac{30}{120}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16}$$

And by symmetry, E is $\frac{1}{2}$. Thus $\frac{P}{E} = 2\left(\frac{7}{16}\right) = \frac{7}{8}$.

(26) E There are multiple ways to place 5 points in such a configuration. One example on the coordinate plane is the points $(0,0), (-1,0), (1,0), (0,1), (0,2)$. Another example is the points $(0,0), (-1,0), (1,0), (-1,1), (1,1)$. Since the configuration is not unique, there is no unique answer to this problem.

(27) B The area in question is equal to double the $60°$ sector minus two equivalent equilateral triangles minus two $120°$ circular segments. The equilateral triangles have side length 3, or area $\frac{9\sqrt{3}}{4}$. Note the small circle has radius $\sqrt{3}$ and the large circle has radius $3\sqrt{3}$ (double the altitude of the equilateral triangle). The circular segments each have area $\pi - \frac{3\sqrt{3}}{4}$. Thus we have $\frac{(3\sqrt{3})^2\pi}{6} - 2\left(\frac{9\sqrt{3}}{4}\right) - 2\left(\pi - \frac{3\sqrt{3}}{4}\right)$ as double the desired area. Thus $2A = \frac{9\pi}{2} - \frac{9\sqrt{3}}{2} - 2\pi + \frac{3\sqrt{3}}{2} = \frac{5\pi}{2} - 3\sqrt{3} \Longrightarrow A = \frac{5\pi}{4} - \frac{3\sqrt{3}}{2}$.

(28) E Let $AC = n$. We are solving for $[ACD] = \frac{n}{2}$. Call X the point where the semicircle is tangent to AD and Y the center of the semicircle. Note that $\triangle DBA \sim \triangle YXA$. We can compute $\frac{BA}{DB} = n + 1$. We can use Pythagorean Theorem to compute $\frac{XA}{YX}$. $XA^2 + \frac{1}{4} = \left(n + \frac{1}{2}\right)^2 \Longrightarrow XA = \sqrt{n^2 - n}$, so $\frac{XA}{YX} = \frac{\sqrt{n^2-n}}{\frac{1}{2}}$. Setting the ratios equal: $\frac{BA}{DB} = n + 1 = \frac{\sqrt{n^2-n}}{\frac{1}{2}} = \frac{XA}{YX}$. Thus $n + 1 = 2\sqrt{n^2 - n}$ and $3n^2 - 2n - 1 = 0 \Longrightarrow n = \frac{1}{3}$ since $n > 0$. Thus $[ACD] = \frac{n}{2} = \frac{1}{6}$.

(29) B It appears that we are missing the side length of the large square, but assigning variables allows us to solve for it! Let A be the upper right vertex of the large square and B be the upper right vertex of the medium square. Let $YA = n$ and by similar triangles with XBY, $AZ = 4n$. Thus the side length of the large square is expressible as $4n$ or $n + 4$ so $n = \frac{4}{3}$.

To solve for the desired area, we subtract out triangles from the area of the whole. In terms of n, the area equals $16n^2 + 16 + 9 - (2n^2 + 2 + \frac{9}{2} + 2n(4n + 7))$, which simplifies to $6n^2 + \frac{37}{2} - 14n$. Plugging in n we get $[WXZ] = 6\left(\frac{16}{9}\right) + \frac{37}{2} - 14\left(\frac{4}{3}\right) = \frac{21}{2}$.

(30) A Geometric series are a wonderful thing. We will use geometric series twice: once for computing the ratio between radii and once to compute the area covered by the circles. Connect any vertex with the center of the inscribed circle, and connect the center with a point of tangency of the inscribed circle to form a $30 - 60 - 90$ right triangle. We see that the radius of the inscribed circle is $\frac{\sqrt{3}}{3}$ and the hypotenuse is $\frac{2\sqrt{3}}{3}$.

It is clear that the radii of the circles form a geometric sequence (draw the line tangent to both circles mentioned in each step of the algorithm to see the pattern more clearly: $X1, Y1, Z1$ are the "inscribed circles" of the generated equilateral triangles).

Thus, the hypotenuse is the same length as the radius of the inscribed circle plus the diameters of the rest of the circles. Let r be the ratio between the radii of the n circle to the $n + 1$ circle: $\frac{2\sqrt{3}}{3} = \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{3}(r + r^2 + r^3 + \cdots) \Longrightarrow r = \frac{1}{3}$.

Note the ratio of the areas of the n circle to the $n + 1$ circle is r^2 by similarity. The area covered by the circles is thus $\frac{1}{3}\pi + 3\left(\frac{1}{3}\pi r^2 + \frac{1}{3}\pi r^4 + \cdots\right) = \frac{1}{3}\pi + \pi\left(\frac{r^2}{1-r^2}\right) \Longrightarrow \frac{11\pi}{24}$.

The area of the full equilateral triangle is $\sqrt{3}$, so the shaded region is $\sqrt{3} - \frac{11\pi}{24}$.