Theta Equations and Inequalities Solutions

1) By using the quadratic formula we obtain the solutions to be $\frac{1\pm\sqrt{8077}}{2}$, hence $A = 1, B = 1, C = 8077, D = 2 \rightarrow A + B + C + D = 8081$ **C**
2) Using difference of squares on each pair of terms we have our sum to be
$$(x-2) + (x-3) + (x-4) + (x-5) + \cdots 1$$
Using the formula for sum of integers we have $\frac{(x-2)\_(x-1)}{2} = 105$
By inspection $x = 16$ is our solution **D**
3) By Heron's formula we can compute the area to be $6\sqrt{6}$
Now notice in the configuration the problem gives the included angle is in between sides of length 7 and 5. Hence $\frac{7*5*\sin BAC}{2} = 6\sqrt{6}$
Solving, we get $\sin BAC = \frac{12\sqrt{6}}{35}$ **B**
4) The shape that will give the maximal area is a circle.
Thus we have $2\pi r = 80 \rightarrow r = \frac{40}{\pi} \rightarrow \pi r^2 = \frac{1600}{\pi}$ **A**
5) Note that absolute values can only sum to 0 when each absolute value is 0 as absolute values are nonnegative so this means
$$x^2 - 3x + 2 = 0, y^2 - 5y + 6 = 0$$
Note that the requested sum is simply $(\sum_{j=1}^{i} x_j)(\sum_{j=1}^{i} y_j)$ so our answer by vietas is (3)(5)=15 **C**
Remark: As the roots were nice we could have just listed the solutions and computed the sum but had the roots been of an uglier form or the equation expanded to multiple variables (such as 5 or 6) the method above would be quicker to use
6) The second equation factors to $(x-1)^2 + (y-2)^2 + 2$. The minimum then occurs at the point (1,2) which trivially satisfies the first equation. Thus the answer is 2 **B**
7) Let m be the number of hours Viraj spends doing math and define s, and c similarly
$m + s + c = 24$-> $\frac{(2)(7)(15)}{4} = \frac{7(9)(8)}{A} \rightarrow A = 9.6$ **D**
8) Notice $x^3 - 3x^2 + x - 3 = (x^2+1)(x-3)$. This motivates us (by Rational Root Theorem) to try the expansion $\big(P(x) - (x-3)\big)\big(P(x) - (x^2+1)\big)$. It turns out that is the correct factorization!. So $P(x) = x - 3$ or $P(x) = x^2 + 1$. Then to possible values are -1003, 1000001 so summing the absolute values gives 1001004 and the sum of the digits is 6 **A**
9) Notice what we are given looks <u>VERY</u> similar to vieta's. Let's list out what we know
$$abc = 16, a + b + c = 4, ab + bc + ac = -4$$

But notice with this information we can construct a cubic with roots a, b, c namely
$$x^3 - 4x^2 - 4x + 16 = (x^2-4)(x-4)$$
So then we only have one distinct value of $a^3 + b^3 + c^3$ and that's 64 **B**

10) $3 - x^2 = 9 - 5x \rightarrow x^2 - 5x + 6 = 0 \rightarrow x = 2,3$ both extraneous **E**

11) The area of the base is just $\frac{8^2\sqrt{3}}{4} = 16\sqrt{3}$ and the height is 3 $V = 48\sqrt{3}$ B
12) Notice we can say based off the conditions $f(x) - x^3 = (x-1)(x-2)(x-3)$.
Now plugging in 4 we have $f(4) = 70$ C
13) For an n×n matrix if we multiply every element by some constant c the determinant of the matrix is multiplied by c^n. Thus our new determinant is $4^3 \times 3 = 192$. Then the sum of digits is clearly 12 D
14) $8^x + 15^x = 17^x \rightarrow \left(\frac{8}{17}\right)^x + \left(\frac{15}{17}\right)^x = 1$. Now if $x < 0$ its equivalent to solving the equation $\left(\frac{17}{8}\right)^a + \left(\frac{15}{8}\right)^a = 1$ or some positive a which clearly is never satisfied. Now on the positive side we see that the left side starts at 2 at $x = 0$ and keeps getting smaller and as $x \rightarrow \infty$ the left side approaches 0. Then there is at most one solution to this equation. By inspection we see the solution at $x = 2$ B
15) The given expression is $-(x-1)^4 + 149$ so max occurs at $x = 1$ B
16) $x = [x^2] - 1980$ implies x is an integer so $x^2$is an integer. Thus we can remove the bars and the problem is equivalent to finding the number of integral solutions to $x^2 - x - 1980 = 0$. This factors to $(x-45)(x+44) = 0 \rightarrow$ **B**
17) $P(x) = (x-a)(x-b)(x-c) \rightarrow -P(2) = (2-a)(2-b)(2-c) = -13$ **D**
18) $a^2 + b^3 + c^2 = (a+b+c)^2 - 2(ab+bc+ac) = 1$.
Substitute $a^2 + b^2 = 1 - c^2$ and similar transforms the expression into $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - 3$
It is known that given any polynomial, if we want to find a polynomial with roots the reciprocal of the previous polynomial, simply flip the coefficients
Hence the polynomial with roots $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ is $x^3 + 12x^2 - 5x + 1 = 0$. A quick application of Vieta's gives the final answer of 151 C
19) Set discriminate equal to 0. $b^2 - 4ac = n^2 - 4(3-2n) = n^2 + 8n - 12 = 0$ use sum of roots to get -8 so A
20) $x \rightarrow -\infty, y \rightarrow -\infty \rightarrow$ **E**
21) $\frac{x^2 - 2019x + 201}{x-1} = \frac{(x-2018)(x-1)+1}{x-1} = x - 2018 + \frac{1}{x-1} \rightarrow x - 2018 = 2019 \rightarrow$ **C**
22) Call $x = a + b$, where $0 \leq b \leq 1$. Equation becomes $a(a+b) = 84 \rightarrow a^2 + ab = 84$ $a^2 + ab < a^2 + a \rightarrow a^2 + a \approx 84, a^2 + a > 84 \rightarrow a = 9 \rightarrow 81 + 9b = 84 \rightarrow b = \frac{1}{3}$ $x = \frac{28}{3} \rightarrow$ **D**

23) The best way to solve this problem is to recall the 3-4-5 and 7-24-25 pythagorean triples. Then the solutions are simply for (a, b)
$(0, \pm 100), (\pm 100, 0), (\pm 60, \pm 80), (\pm 80, \pm 60), (\pm 28, \pm 96), (\pm 96, \pm 28)$ D
24) Let a, b, c, d be the roots of the first polynomial. Observe than that the roots of the 2nd polynomial are $\ln a, \ln b, \ln c, \ln d \rightarrow \ln a + \ln b + \ln c + \ln d = \ln abcd = \ln 4$ A
25) We are told that there is a unique positive unique integral solution, is it's not a bad idea just to plug in the solutions and see $x = 4$ works C

26) Note the equation looks similar to Heron's formula. In fact if A is the area of a triangle with side lengths x, y, z then the given equation is $16A^2$ so $m = 16A^2 \rightarrow \sqrt{m} = 4A$
But then observe the triange is right with legs of $\sqrt{5}, 2\sqrt{2}$. Hence it remains to compute the floor of $4\sqrt{10}$ which is 12 B

27) Using the quadratic formula we require $a^2 \geq 4b$. Now we can proceed with cases based off of a. $a = 0, b = 0, a = 1, b = 0, a = 2, b = 0{,}1, a = 3, b = 0 \rightarrow 2, a = 4, b = 0 \rightarrow 4, a = 5, b = 0 \rightarrow 6, a = 6, b = 0 \rightarrow 9, a = 7 \rightarrow 10, b = 0 \rightarrow 10$. Hence there are 73 ordered pairs C

28) Harmonic mean is $\frac{2(40(60)}{40+60} = 48$. The arithmetic mean is obviously 50. Then the the positive difference is simply 2 B

29) Dividing by 4 we see the second equation is equivalent to the first. Then the equations are dependent since they are the same line and consistent as there is more than one solution to the system. D

30) By the fundamental theorem of algebra we are seeking for the degree of the polynomial which is clearly 9 A