2019 Mu Alpha Theta National Convention - Theta Logarithms & Exponents Solutions

1. D
2. B
3. A
4. D
5. C
6. E
7. B
8. A
9. D
10. E
11. A
12. C
13. D
14. B
15. D
16. B
17. B
18. D
19. A
20. B
21. C
22. D
23. A
24. B
25. C
26. C
27. D
28. C
29. E
30. A

1) $\log_1 1$ is undefined.

2) $\log_2(\log_2 x + \log_4(x^2)) = \log_2(2\log_2 x)$. Thus, $2\log_2 x = 2^{2019}$, so $x = 2^{2^{2018}}$.

3) $\log_{0.03125} 0.25 = \log_{0.5^5} 0.5^2 = \frac{2}{5}\log_{0.5} 0.5 = \frac{2}{5}$.

4) $72^3 = (2^3 3^2)^3 = 2^9 3^6$. This number has $(9+1)(6+1) = 70$ factors.

5) $\log_{84} 300 = \frac{\log_2 300}{\log_2 84}$ by change of base. This equals $\frac{\log_2 4 + \log_2 3 + \log_2 25}{\log_2 4 + \log_2 3 + \log_2 7} = \frac{2+\alpha+2\beta}{2+\alpha+\gamma}$.

6) First, $18 - 3x - x^2 = (6+x)(3-x) > 0$. This has the solution set $(-6,3)$. Second, $x^2 + 2x - 8 = (x+4)(x-2) > 0$. This has the solution set $(-\infty,-4) \cup (2,\infty)$. Also, $x \neq 0$ from the $2019^{x^{-2}}$ term. The intersection of these ranges is $(-6,-4) \cup (2,3)$. However, the base of a logarithm cannot be 1, so $x = \frac{-3\pm\sqrt{77}}{2}$ must be excluded.

7) $(1+i)^{2019}(1-i) = (1+i)^{2018}(1+i)(1-i) = (2i)^{1009}(1-i^2) = 2^{1009}i^{1009}2 = 2^{1010}i$.

8) The determinant of the matrix is $6\ln x + 20\ln^2 x - 2\ln x - 45\ln^2 x = -25\ln^2 x + 4\ln x = \ln x\,(4 - 25\ln x)$. Thus, $x = 1$ or $x = e^{4/25}$. The product of these solutions is $e^{4/25}$.

9) The number of complex solutions to the equation is k.

10) Setting $x = \sqrt{n + \sqrt{n + \sqrt{n + \cdots}}}$, the simplification $x = \sqrt{n + x}$ can be made. Squaring and solving for x, this is equal to $\frac{1+\sqrt{1+4n}}{2}$. Thus, $1 + 4n$ must be an odd square. The greatest odd square that would correspond to a value of $n \leq 200$ is $729 = 27^2$ $(n = 182)$. There are 14 odd squares less than or equal to 729. However, 1^2 $(n = 0)$ is an extraneous solution when solving the equation for $n = 0$. Thus, there are 13 solutions.

11) Substituting $y = \ln x$, the equation becomes $y^2 + 8y + 4 = 0$. By Vieta's, this has a sum of solutions of -8 (note that both solutions are real). The sum of the solutions of the translated equation can be converted to the product of the solutions of the original equation by exponentiation, so the product of the solutions of the original equation is e^{-8}.

12) $2019 \equiv 3 \bmod 4$, and $9 \equiv 1 \bmod 4$. Thus, the given expression has an identical last digit to $2^3 + 0^3 + 1^3 + 9^3 + 9^2 + 9^0 + 9^1 + 9^1$. The last digit of this modified expression is equal to $(8+0+1+9+1+1+9+9) \equiv 38 \bmod 10$. Thus, the last digit of the expression is 8.

13) This is change of base for $\log_a(\log_b a)$. a to the power of this is $\log_b a$.

14) $\log_2 2^{3^4} = 3^4 \log_2 2 = 3^4 = 81$.

15) 12^8 is 1 more than a multiple of 11. Using the divisibility test for 11, the sum of the digits in odd indices in the number minus 1 is $4+9+8+6+5 = 32$, and the sum of the digits in even indices in the number minus 1 is $2 + X + 1 + 9 = 12 + X$. The difference of these values is $20 - X$, which must be a multiple of 11. $X = 9$ is the only possible solution.

16) $\frac{x^{2a+b+2}y^{3a+4b-5}}{y^{-4a+6b+1}x^{3a-3b-2}} \div \frac{x^{4a+2b}y^{5a-2b+4}}{x^{-2a-6b-1}y^{-a+b+3}} = \frac{x^{-a+4b+4}y^{7a-2b-6}}{x^{6a+8b+1}y^{6a-3b+1}} = x^{-7a-4b+3}y^{a+b-7}$. $R = -7, S = -4, T = 3, U = 1, V = 1$, and $W = -7$, so $RU + SV + TW = -32$.

17) By Euler's Theorem, $7^{\varphi(2019)} \equiv 1 \bmod 2019$. $\varphi(2019) = 2019\left(1-\frac{1}{3}\right)\left(1-\frac{1}{673}\right) = 2 \cdot 672 = 1344$. Thus, the expression is equivalent to $7^6 \bmod 2019 \equiv 547 \bmod 2019$.

18) $\log_{16} 27 \cdot \log_{49} 512 \cdot \log_{625} 343 \cdot \log_{729} 15625 = \log_{16} 512 \cdot \log_{729} 27 \cdot \log_{625} 15625 \cdot \log_{49} 343 = \frac{9}{4} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{81}{32}$. $H = 81$ and $K = 32$, so $H + K = 113$.

19) Continuously compounding interest makes the amount of money Trevor has equal to $1337e^{0.06}$ after t years.

20) Start with the case $x^2 - 7x + 11 = 1$. This simplifies to $x^2 - 7x + 10 = 0$, so $x = 2$ or $x = 5$. The case $x^2 + 4x - 45$ has solutions $x = -9$ and $x = 5$. Finally, the case where $x^2 - 7x + 11 = -1$ and $x^2 + 4x - 45$ is even simplifies to $x^2 - 7x + 12 = 0$ and x must be odd. This has the solution $x = 3$, and the solution $x = 4$ is extraneous. Thus, the solution set for x is $\{-9,2,3,5\}$. The sum of the elements of this set is 1.

21) Due to the exponential nature of the function, it can be found that $f(0) = a = 12$. Solving $12b^3 = 6$ and $12b^6 = 3$ both yield $b = 2^{-1/3}$. $2^{-(-1/3)\cdot 12} = 2^4 = 16$.

22) $5^{500}2^{1000} = 2^{500}10^{500}$. This number is 2^{500} with 500 zeroes appended to the end of it. The number of zeroes in 2^{500} is $1 + \lfloor 500 \log_{10} 2 \rfloor = 1 + \lfloor 500 \cdot 0.301 \rfloor = 1 + 150 = 151$. This plus 500 zeroes is 651 total digits.

23) The sum is equal to $1(12-1) + 2(144-12) + 3(1728-144) + 4(2019-1728) = 6191$. This lies in the range $[6100,6199]$.

24) Note that $\frac{2}{3} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$. This is equal to $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$. This is in $0.\overline{10}$ binary.

25) The expression is equivalent to $3^{6x+14} = 3^{10x-5}$. Solving $6x + 14 = 10x - 5$ yields $x = \frac{19}{4}$. $A = 19$ and $B = 4$, so $A + B = 23$.

26) Examining $e^{1/x} > 1$ yields $\frac{1}{x} > 0$, or $x \in (0, \infty)$.

27) The probability Eridan flips the first head is equal to the probability he flips a head on his first flip, plus the probability he flips a head after he and Terezi both flip tails once, plus the probability he flips a head after he and Terezi both flip tails twice, and so on. This is equal to $\frac{1}{3} + \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + \cdots = \frac{\frac{1}{3}}{1-\frac{4}{9}} = \frac{\frac{1}{3}}{\frac{5}{9}} = \frac{3}{5}$.

28) The characteristic equation for the recursion is $x^2 = 9x - 20$. This has solutions $x = 4$ and $x = 5$, so $x_n = A_1 4^n + A_2 5^n$. Plugging in $n = 0$ and $n = 1$ yields the system of equations $A_1 + A_2 = 10$ and $4A_1 + 5A_2 = 47$, so $A_1 = 3$ and $A_2 = 7$. It is already known that $B_1 = 4$ and $B_2 = 5$, so $A_1 + A_2 + B_1 + B_2 = 19$.

29) The expression is equal to $\log_{2019} 2 \cdot \log_{2019} 3 \cdot \ldots \cdot \log_{2019} 2019$. 2019 to the power of this is not any clean closed form.

30) (1) passes because the positive reals are closed under multiplication. (2) is the Distributive Property for multiplication. (3) is the Associative Property for multiplication. (4) passes, given the vector $\mathbf{0} = 1$. (5) passes, given the vector $-\boldsymbol{u} = \frac{1}{\boldsymbol{u}}$. (6) passes because the positive reals are closed under

exponentiation. (7) is the identity $(uv)^c = u^c v^c$. (8) is the identity $u^{c+d} = u^c u^d$. (9) is the identity $u^{cd} = (u^c)^d$. (10) is the identity $u^1 = u$.

None of the axioms fail, and V is confirmed to be a vector space.