Expansion/Analysis of a Card Trick Comprised of Transformations in 2-Dimensional Matrices

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This paper illustrates the properties of a card trick which is commonly called, "The 11th Variation." The card trick, which involves transformations in 2-dimensional matrices, is also expanded and explained with variations in this paper. Certain predictions as to the parameters that govern the outcome of the trick are made. Possible application is also suggested.

The original card trick ("The 11th Variation") is merely the foundation of this paper. The classic example of this card trick employs 21 cards and requires 3 rearrangements of the cards. The goal of the card trick is to "guess" which of the cards a person is thinking of. After some repeated rearrangement of the cards, no matter where in the matrix, the card always ends up as the 11th card or the middle card in the matrix.

In order to find out how the trick worked the standard card trick was examined. First, 21 cards (as in the standard card trick) are laid out into 3 columns of 7 cards in the following order:

1	**8**	15
2	9	16
3	10	17
4	11	18
5	12	19
6	13	20
7	14	21

A person is asked to choose any one of the 21 cards and tell you only which column it is in. In this example, the chosen card is 8. Each of the columns is collected *in order* into three separate piles so that *the bottom card in each column is on top and the top card in each column is on the bottom*. The piles are stacked so that the pile with the "chosen column" (which contains the chosen card) is placed between the other two. The deck is turned upside down so that the back of the cards are showing. The cards in the stack (which were previously arranged in columns) are rearranged in rows. The cards should now be juxtaposed in the following fashion:

1	2	3
4	5	6
7	**8**	9
10	11	12
13	14	15
16	17	18
19	20	21

Once again, the person is asked to find their chosen card and only tell you the location in terms of which column it is in. The cards are collected in the same order and fashion as before. The new "chosen column" (containing the chosen card) is placed in a pile between the other two piles and the deck is flipped (the same procedure as before). Rearranging the cards in rows again will produce the following layout:

1	4	7
10	13	16
19	2	5
8	11	14
17	20	3
6	9	12
15	18	21

Again, the person is asked which column is the "chosen column." Column 1 is placed between 2 and 3, the deck is flipped, and the cards are laid out in rows. Now, the middle card (11th card) in the matrix is the chosen card.

4	13	2
11	20	9
18	1	10
19	**8**	17
6	15	7
16	5	14
3	12	21

Note that in the previous steps, the chosen card progressively moved toward the middle. In this example (7 row by 3 column configuration), the person conducting the trick must ask for the "chosen column" at least three times to be sure that the chosen card is in the middle of the matrix. I proved this by trying the trick with each of the 21 cards. Depending on their position, the cards required asking numbers from 1 to 3 to arrive at the center position. Once achieved, the chosen card will remain at the middle position no matter how many times the conductor asks. A minimum of 3 rearrangements is required to guarantee the chosen card to be the 11th card (middle card).

Four important questions come to mind that serve as a basis for expansion and variation of the trick:

1. Is it possible for this trick to work with more or less than 21 cards?
2. Is it possible to make the card appear in another position of the matrix?
3. Is it possible to use an even number of rows and columns?

Variable/ Term Definitions:

- Asking Number or Number of Askings (variable = *n*) - the minimum amount of times the person conducting the trick is required to ask for the "chosen column" in order to be sure that the chosen card is in the middle position.
- Column Number (variable = *c*) - the amount of columns in the matrix.
- Row Number (variable = *r*) - the amount of rows in the matrix.
- Card Number (variable = *k*) - the amount of total cards in the matrix (thus, *k* = *rc*).
- Asking Set (variable = *s*) - the number of possible combinations of rows (with a constant number of columns) that have the same asking number.
- Card Number Exponent (variable = *y*) - the exponent that will make a given number of columns equal to a given card number, i.e. $c^y = k$.

Observations and Expansion

Asking Number Variations

After trying the card trick with several different arrangements of cards, I realized that the asking number varies according to the number of rows and columns. In order to establish some type of generalization, I looked for patterns in the asking numbers and rows when *c* was kept constant at 3 (*c* = 3 in the original trick).

An example of one of my trials is shown below using an 11 row by 3 column array. I noticed that the asking number is 4 rather than 3 (in the original trick).

	0 = *n*			1 = *n*			2 = *n*			3 = *n*			4 = *n*	
1	**12**	23	1	2	3	1	4	7	1	10	19	10	6	33
2	13	24	4	5	6	10	13	16	28	6	15	26	25	21
3	14	25	7	8	9	19	22	25	24	33	8	14	4	31
4	15	26	10	11	**12**	28	31	3	17	26	7	27	20	1
5	16	27	13	14	15	6	9	**12**	16	25	3	28	24	17
6	17	28	16	17	18	15	18	21	**12**	21	30	16	**12**	5
7	18	29	19	20	21	24	27	30	5	14	23	32	22	18
8	19	30	22	23	24	33	2	5	32	4	13	11	19	15
9	20	31	25	26	27	8	11	14	22	31	9	8	7	3
10	21	32	28	29	30	17	20	23	18	27	2	30	23	13
11	22	33	31	32	33	26	29	32	11	20	29	9	2	29

The results of the other trials are shown below:

k (= rc)	*r*	*n*	*s*
3	1	1	NA
9	3	2	1 = 3^0
15	5	3	
21	7	3	3 = 3^1
27	9	3	
33	11	4	
39	13	4	
45	15	4	
51	17	4	
57	19	4	9 = 3^2
63	21	4	
69	23	4	
75	25	4	
81	27	4	
87	29	5	
93	31	5	
99	33	5	
105	35	5	
111	37	5	
117	39	5	27 = 3^3
123	41	5	
129	43	5	
135	45	5	
↓	↓	↓	
243	81	5	
249	83	6	81 = 3^4
↓	↓	↓	

729	243	6	(predicted)
735	245	7	
↓	↓	↓	

In order to test such a large number of cards, I used five decks. (261 cards is the highest possible divisible by 3 (with the joker). The joker was used as a test chosen card). Therefore, I was able to test 3 columns up to 87 rows. The rest of the table was predicted according to the asking number pattern which is set by rows 1-87. The pattern concerns the asking set value (*s*). For example, in the table above, when $c = 3$ and $r = 11 \ldots 27$, then $s = 9$. There are 9 possible combinations for which the asking number is 3. The number 9 is significant because it is 3^2. Note that 3 is the number of columns and the exponent (which is 2 in this example) is $n - 2$. This pattern is consistent for the other asking sets (see table). Therefore the following equation can be formed concerning asking sets:

$$s = c^{n-2}$$

To see if this equation would remain constant for other scenarios with a different number of columns, c=5 was tested. For the next set of trials, I skipped c=4. (The reason I skipped 4 columns is because there cannot be a unique position that is both the vertical and horizontal median when using an even number of rows or columns.) Results when c=5:

k (= rc)	*r*	*n*	*s*
5	1	1	NA
15	3	2	$2 = 5^0 x2$
25	5	2	
35	7	3	$10 = 5^1 x2$
45	9	3	
55	11	3	
65	13	3	
75	15	3	
85	17	3	
95	19	3	
105	21	3	
115	23	3	
125	25	3	
135	27	4	$50 = 5^2 x2$ (predicted)
↓	↓	↓	
625	125	4	
625	127	5	$250 = 5^3 x2$ (predicted)
↓	↓	↓	
3125	625	5	
3135	627	6	
↓	↓	↓	

As you can see, the card numbers increase much faster in this chart. I was not able to discover enough of the table with 261 cards this time (I tested up to 255 cards in this case). However, the pattern from the first table with c=3, helped me predict the rest of the n values for $c = 5$. As in the $c = 3$ chart, the greatest number of cards in a series is equal to the greatest number of rows in the next series. By the means of this method, I was able to predict that the $n = 5$ asking series ends at 625. Using this information, I calculated the asking set equation. The ratio of the difference in *r* for each asking set to the asking set value is 2:1. For example:

$$125 - 25 = 100$$
$$100/2 = 50 = 5^2 x2.$$

Also:

$$625 - 125 = 500$$
$$500/2 = 250 = 5^3 x2.$$

The asking set equation for c = 5 is:

$$s = 5^{n-2} x2$$

At this point I had guessed that the asking set equation *c* = 7. I renamed the asking set equation for *c* = 3 as $s = 3^{n-2}x1$ rather than $s=3^{n-2}$, so that it is easier for comparison. My predictions are shown below:

$$(c = 3)\ s=3^{n-2}x1$$
$$(c = 5)\ s=5^{n-2}x2$$
$$(c = 7)\ s=7^{n-2}x3$$
$$(c = 9)\ s=9^{n-2}x4$$

To justify these predictions, I charted the results for *c* = 7 and *c* = 9. In order to find the value of the asking set, I used the same method that I used for *c* = 5. The method, once again: As in the *c* = 3 chart, the greatest number of cards in a series is equal to the greatest number of rows in the next series. Charts for c = 7 and c = 9 can be found in the printable version of this essay.

After completing the four tables (*c* = 3, 5, 7, 9) I observed that:

- $s = c^{n-2}(c - 1)/2$
- The highest possible *r* in an asking set is c^{n-1}.
- The charts serve as a quick and easy reference to find the required asking number when the number of columns and rows is known.

There are many other mathematical patterns and applications that can be created from the charts.

Different Arrangements in 2 Dimensions

In the original "11th Variation" trick, the card always appears in the middle of the matrix after a given number of askings. The card always appears in the middle because the "chosen column" is placed as the middle pile. I decided to place the "chosen column" on the bottom of the stack. After repeating this 3 times with the 3 by 7 configuration, the card appeared in the upper left corner of the matrix. Then, I placed the "chosen column" on top of the stack. After repeating this 3 times with the *c* = 3/ *r* = 7 configuration, the card appeared in the lower right corner of the matrix.

The advantage of making the cards appear in the corners is so that the trick can be performed with an even number of columns and/or rows. The conductor can use even and even or even and odd arrangements such as 4x4 or 3x4. Below is an example of a trick which utilizes an even number arrangement (4x4).

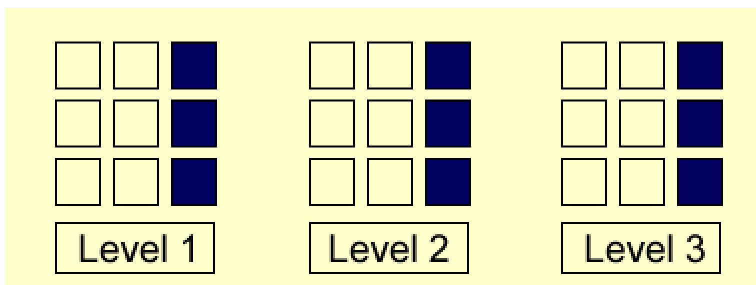
$n = 0$			
1	5	9	13
2	**6**	10	14
3	7	11	15
4	8	12	16

$n = 1$			
1	2	3	4
9	10	11	12
13	14	15	16
5	**6**	7	8

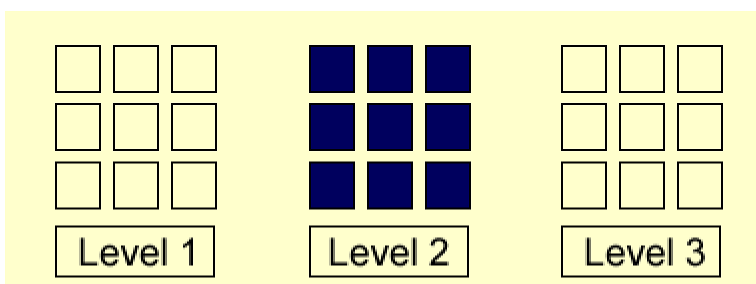
$n = 2$			
1	9	13	5
3	11	15	7
4	12	16	8
2	10	14	**6**

3 Dimensional Matrices

The 3 dimensional aspects of similar rearrangements can also be researched. To get a card to the center of a 3 dimensional matrix, the number of rows, columns, and levels must be odd. An example of a 3 dimensional matrix is a 3x3x3 arrangement. To perform the trick in three dimensions, the conductor must ask for the "common elevated column" which is shaded in the diagram below.



In this example, there are nine cards shaded. Each of the nine cards will be placed in a pile between the other two "common elevated columns." The conductor will lay out the cards by each level in rows, so that the arrangement will look like the one shown below.



The conductor will continue this process for a certain amount of asking numbers, until the card appears in the middle layer in the middle of the matrix.

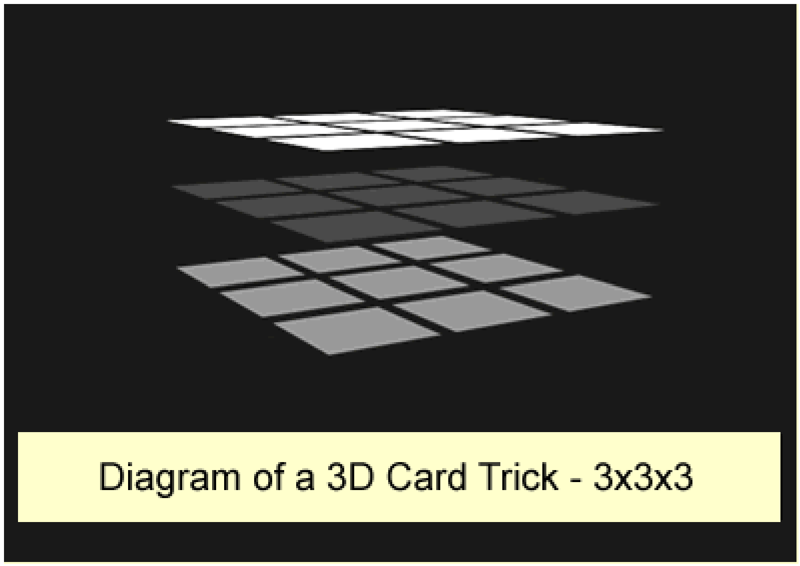
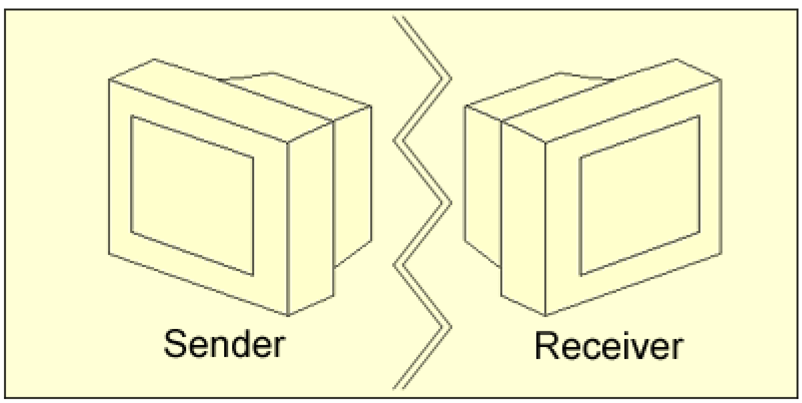


Diagram of a 3D Card Trick - 3x3x3

Application

The concept of this card trick can be applied to encrypt information in interactive communication between two or more people. The ideal circumstances for this application are two people communicating through an interactive website. A diagram of an ideal situation is shown below:



The "sender" is the person who knows the "chosen card" and wants to securely notify the "receiver" which is the "chosen card." Each card has a unique action. The receiver will carry out the specified action on the "chosen card" when he/she finds out the location of the card in the matrix.
The people will first decide on the layout (rows and columns). The sender will begin by telling the receiver which column the card is in by clicking any random card in the "chosen column." Then the receiver will rearrange the cards accordingly. This process will continue for the number of times necessary (the asking number for the chosen layout) to bring the card into the middle of the matrix, or the corners of the matrix.

Any outsider or bystander who is watching will not be able to extract information because the card that is being clicked by the sender is not the chosen card, but rather a card in the "chosen column." The bystander will see the changing matrix because only the "chosen column" is required by the receiver and specified by the sender.

The variations of the trick that are discussed in this paper can be used to produce an efficient and secure fashion to transfer ideas between people.

Conclusion

This paper explores some of the variations of a card trick. Mathematical relationships can be developed to predict how the variations would work. Different arrangements of cards should reveal more patterns and mathematical relationships. Although the card trick is introduced as "The 11th Variation," there are numerous permutations and variations that can be formed by changing certain properties, such as the number of columns and rows. The mathematical analysis in this paper allows for prediction of the parameters that are necessary to predict the chosen card under different circumstances. The results of this paper can be used to produce an interactive encryption system.

Additional findings can be found in the printable version of this article.

(Editor's note: Readers are challenged to prove the patterns Aaron identifies in this article. Further, readers are encouraged to mathematically analyze and generalize other card tricks they know to see if they can prove why the card tricks work.)