Exploring Moduli Spaces

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When most people think "math" they think "numbers" or "equations." But much of mathematics concerns space--not outer space, but mental representations of space and of spatial relationships.

While many people think of Descartes as the first person to describe points in a plane by means of a pair of coordinates, the idea of coordinatizing planar space was known to Hipparchus in 150 B.C. And by 1640, Pierre de Fermat had already developed the idea of expressing a curve in the plane as the solution set of an equation in x and y. René Descartes, however, was the first to systematically use the coordinate plane to turn problems of geometry into problems of algebra, and vice versa. Today, at the very center of modern research in number theory and geometry lies a much-generalized form of Descartes's idea--that of moduli spaces.

What is a Moduli Space?

You might think of a moduli space as a map or a picture of a certain set. A map of the United States is, loosely speaking, a moduli space for the set of states. If you had a list of the 50 states and their sizes and shapes, and another list detailing which states bordered on which other states, you might still have only a fuzzy idea of what the United States looks like and how it is put together. But with a map, you would perceive this information clearly.

In math, you typically work with sets of objects, such as numbers, points, or lines. But just as with the states, you can often have a clearer understanding of a set if you have a "map" of it. A moduli space for a mathematical set might be a plane, a line, a curve, a cylinder, a hockey-puck-shaped entity, or even a wild hyperamoeba in 26-dimensional space. More specifically, a moduli space for a set S is a geometric object X such that

1) the points of X are in one to one correspondence with the elements of S; and

2) elements of S that are close to each other correspond to points of X that are close to each other, and vice versa.

The Coordinate Plane

Just as a picture of the United States is a map of the set of the states, the plane is a map of the set of pairs of real numbers. If S is the set of ordered pairs (x,y) of real numbers, a moduli space for S would be a plane, which we'll call X.

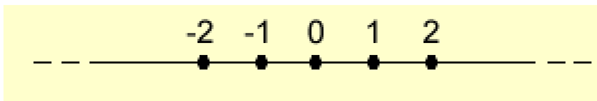
How does X satisfy our definition of moduli spaces? The one to one correspondence required by the first part of the definition sends an ordered pair (x,y) to the point (x,y) of the plane. The second part of the definition is satisfied because pairs of real numbers that are very close to each other, such as (1,2) and (1.001, 1.999), map to points of *X* that are very close to each other; likewise, nearby points of *X* correspond to nearby pairs of real numbers.

So Descartes's insights about the coordinate plane can be summed up in the terminology of moduli spaces: A plane is a moduli space for the set of pairs of real numbers.

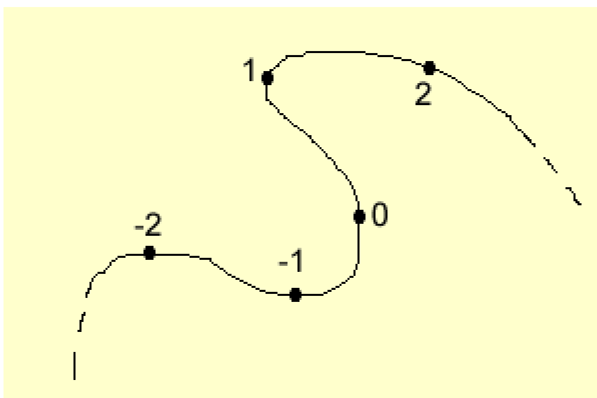
Challenge 1: What would be a moduli space for the set of unordered pairs of real numbers (*x,y*)?

Distorting Space

You can probably guess that a line is a moduli space for the set of all real numbers.



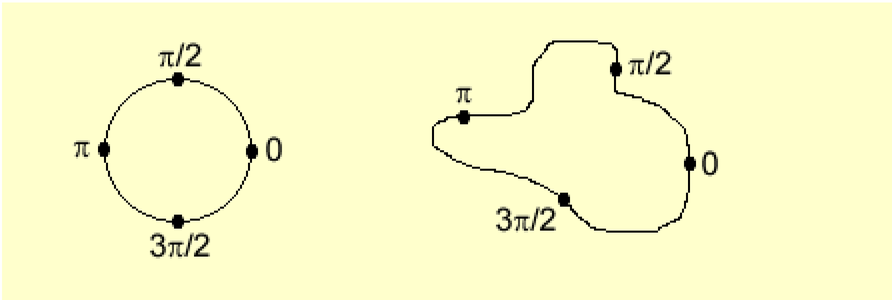
But it is not the only one. We can distort the line in as devious a fashion as we like, taking care not to break it or let it cross itself (which would violate the second and first parts of our definition, respectively) and the resulting curve will still be a moduli space for the real numbers.



This example enables us to expand our definition of moduli spaces a bit: If *X* is a moduli space for a set *S*, any distortion of *X* is also a moduli space for *S*.

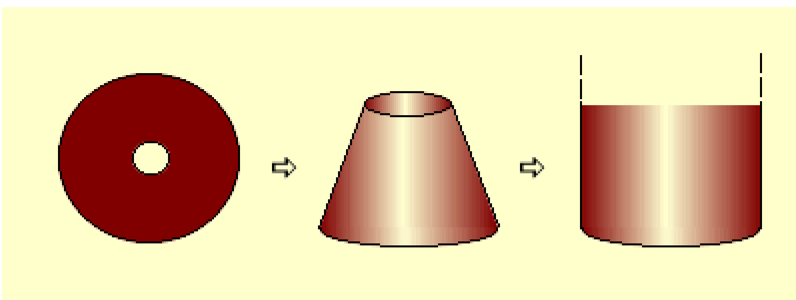
Circles, Cylinders, and More Moduli Spaces

What if *S* is the set of all angles? We know that an angle is determined by its measure, which (in radians) is a real number between 0 and 2π. But the moduli space of *S* is not a line segment from 0 to 2π because those points, which are far away from one another in a line, correspond to angles that are identical to each other, violating our definition. To make a moduli space for *S*, we need to join the ends of the line segment to form a circle. Of course, a distorted circle will also do.



Now let *S* be the set of ordered pairs (r, θ), where *r* is a positive real number and θ is an angle. If you know about polar coordinates, you know that the plane with the origin removed is a moduli space for *S*: (r, θ) corresponds to the point whose (x,y) coordinates are $(r \cos \theta, r \sin \theta)$. This moduli space is usually called "the punctured plane."

We can also think of the moduli space for this set in another way. We know that a circle is a moduli space for the set of angles. And we can see that a ray is a moduli space for the set of positive real numbers. So a moduli space for *S* can be constructed by starting with a circle and attaching a ray to each point. Of course, we don't have to plot *r* in the same plane as θ. If *r* is plotted perpendicular to the *xy* plane, the moduli space now looks like a cylinder, infinite in one direction. Note that this moduli space is just a distortion of the punctured plane.



Challenge 2: What if *S* is the set of ordered pairs (r, θ), where *r* is any real number?

Challenge 3: What if S is the set of ordered pairs of angles (θ_1, θ_2)?

Playing Around with Correspondences

Suppose *S* is the set of all lines through the origin of the xy plane. What is a moduli space for *S*?

A line through the origin can be defined by the angle θ it makes with the *x* axis. So we might guess that the moduli space for *S* is the same as the moduli space for angles: a circle. But as we consider the problem more carefully, we hit a snag. The natural correspondence between lines and angles is not one to one, but two to one: the angles π /2 and 3π /2 both correspond to the vertical line. How can we address this problem?

One good way is to use a different correspondence between lines and angles: Instead of associating line *L* to the angle θ, we could associate it to the angle 2 θ. That simple redefinition transforms our two to one correspondence to a one to one correspondence. Using this new correspondence, we can see that the moduli space for *S* is indeed a circle.

Challenge 4: What if *S* is the set of all lines in the *xy* plane?

What I like about problems like this one is that they don't require much, or any, pencil and paper calculation. You can work on them in the shower or while riding the bus. Better yet, the problem admits endless variations, each with its own interesting features: What is the moduli space of circles in a plane? Circles through the origin? Lines through the origin in three dimensional space? The possibilities are limited only by your imagination (and, if you are working in the shower, by the hot water supply.) Enjoy!

Answers to challenges

Challenge 1: *X* is a half-plane: for example, the half-plane defined by $\{(x,y): x \geq y.\}$
Challenge 2: *X* is an infinite cylinder.
Challenge 3: *X* is shaped like the surface of a doughnut - a surface mathematicians call a "torus."
Challenge 4: *X* is the famous Möbius strip.

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