Mu Ciphering Nationals 2021 Solutions

Answers:

0. y=2x
1. $\frac{2}{3}$
2. $2\sqrt{89}$
3. $\frac{35}{6}\pi$
4. $72\sqrt{5}$
5. $\frac{\pi}{4}$
6. $\frac{4}{5}$
7. $\frac{5\sqrt{3}}{2}-\frac{\pi}{3}=\frac{15\sqrt{3}-2\pi}{6}$
8. $\frac{1}{10}$
9. $36\sqrt{3}$
10. 4
11. $\frac{32}{15}$
12. $\frac{L+U+3}{L+U}$

0. $y' = \cos x + 1 \rightarrow y' = 2 \rightarrow y = 2x$

$$A = \int_k^1 3x^2 dx + \int_1^{k+2} (4-x)dx \rightarrow x^3 \Big|_k^1 + 4x - \frac{x^2}{2} \Big|_1^{k+2}$$

1. $1 - k^3 + 4k + 8 - \frac{k^2+4k+4}{2} - \frac{7}{2} = -k^3 - \frac{k^2}{2} + 2k - \frac{9}{2}$

$$A' = -3k^2 - k + 2 = (k+1)(2-3k) = 0 \rightarrow k = \frac{2}{3}$$

2.

$$\cos J = \pm\frac{2\sqrt{2}}{3} \rightarrow (PF)^2 = 9 + 25 - 2 \bullet 3 \bullet 5 \bullet \left(\pm\frac{2\sqrt{2}}{3}\right)$$

$$34 \pm 20\sqrt{2} \rightarrow PF = \sqrt{\left(34+20\sqrt{2}\right)\left(34-20\sqrt{2}\right)} = \sqrt{356} = 2\sqrt{89}$$

3. Use shell method

$$2 - x^2 = 3x - 2 \rightarrow x^2 + 3x - 4 = 0 \rightarrow x = \{-4, 1\}$$

$$2\pi\int_0^1 (x+1)\left[2 - x^2 - (3x-2)\right]dx = 2\pi\int_0^1 (x+1)\left(-x^2 - 3x + 4\right)dx$$

$$2\pi\int_0^1 \left(-x^3 - 4x^2 + x + 4\right)dx \rightarrow 2\pi\left(\frac{-x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} + 4x\right)\Big|_0^1 \rightarrow 2\pi\left(\frac{-1}{4} - \frac{4}{3} + \frac{1}{2} + 4\right)$$

$$2\pi\left(\frac{-3-16+6+48}{12}\right) = \frac{35}{6}\pi$$

$$A=\frac{1}{2}bh=\frac{1}{2}\left(\frac{2b^2}{a}\right)(2c)=\frac{2b^2c}{a}$$

4.

$$x^2-4y^2+10x+24y+25=0\to(x-5)^2-4\left(y^2-6y+9\right)=-36$$

$$\frac{(y-3)^2}{9}-\frac{(x-5)^2}{36}=1\to a=3\to b^2=36\to c^2=45\to c=3\sqrt{5}$$

$$\frac{2b^2c}{a}=\frac{2\bullet 36\bullet 3\sqrt{5}}{3}=72\sqrt{5}$$

5. $\int_{0.5}^{1}\frac{dx}{2\sqrt{x-x^2}}=\int\frac{dx}{2\sqrt{x}\sqrt{1-x}}$. Let u= $\sqrt{x}$

and $du=\frac{1}{2\sqrt{x}}$ so we have $\int\frac{1}{\sqrt{1-u^2}}du$

which gives $Arc\sin\sqrt{x}+c$ evaluated

from x=0.5 to x=1. Arcsin1-Arcsin $\frac{\sqrt{2}}{2}$

gives $\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$.

6. Draw a good picture. Bisect angle XZY and call the point where it hits side XY “L”. This makes WXLZ a parallelogram so XL=5x and LY=4x. Then use angle bisector theorem on triangle XZY

$$\frac{1}{5x}=\frac{ZY}{4x}\to ZY=\frac{4}{5}$$

$$\frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}}(1+2\cos x)^2\,dx-\frac{4\pi}{3}=\int_0^{\frac{\pi}{3}}(1+2\cos x)^2\,dx-\frac{4\pi}{3}=\int_0^{\frac{\pi}{3}}\left(1+4\cos x+4\cos^2 x\right)dx-\frac{4\pi}{3}$$

7. $$\int_0^{\frac{\pi}{3}}(1+4\cos x+2+2\cos 2x)\,dx-\frac{4\pi}{3}=\int_0^{\frac{\pi}{3}}(3+4\cos x+2\cos 2x)\,dx-\frac{4\pi}{3}$$

$$4\sin x+3x+\sin 2x\Big|_0^{\frac{\pi}{3}}-\frac{4\pi}{3}=2\sqrt{3}+\pi+\frac{\sqrt{3}}{2}-\frac{4\pi}{3}=\frac{5\sqrt{3}}{2}-\frac{\pi}{3}=\frac{15\sqrt{3}-2\pi}{6}$$

8. $5\bullet 4\bullet 3\bullet 2\bullet 1=120$ so we have 120 possibilities. The given numbers sun to 24. To be divisible by 11 the sum of the first, third and fifth digits minus the sum of the 2nd and 4th digit must be a multiple of 11. The only way to make this work is if both sets sum to 12. So, 1st, 3rd, and 5th digits must be 7,3, and 2 is some order. The 2nd and 4th digits must be 8 and 4 is some order. $$3\bullet 2\bullet 2\bullet 1\bullet 1=12$$ $$\frac{12}{120}=\frac{1}{10}$$

9. The area of a hexagon is $A=\frac{3s^2\sqrt{3}}{2}$. The longest diagonal is twice the side length. $y=\frac{\sqrt{36-9x^2}}{2}$ represents half the diagonal which is the side of the hexagon

$$V=\frac{3\sqrt{3}}{2}\int_{-2}^{2}\frac{36-9x^2}{4}dx=3\sqrt{3}\int_0^2\left(9-\frac{9x^2}{4}\right)dx=3\sqrt{3}\left(9x-\frac{3x^3}{4}\right)\Big|_0^2$$

$$3\sqrt{3}(18-6)=36\sqrt{3}$$

10.

$$\log_4 k+\log_{k^2}\frac{1}{8}=1\rightarrow\frac{1}{2}\log_2 k-3\log_{k^2}2=1$$

$$\frac{1}{2}\log_2 k-\frac{3}{2}\log_k 2=1\rightarrow\log_2 k-\frac{3}{\log_2 k}=2$$

$$\log_2 k=x\rightarrow x-\frac{3}{x}=2\rightarrow x^2-2x-3=0\rightarrow x=3,-1$$

$$k=8,\frac{1}{2}\rightarrow 8\bullet\frac{1}{2}=4$$

11.

$$2\pi\int_0^{2p}\left(p-\frac{x^2}{4p}\right)^2 dx = 2\pi\int_0^{2p}\left(p^2-\frac{x^2}{2}+\frac{x^4}{16p^2}\right)dx$$

$$2\pi\left(p^2x-\frac{x^3}{6}+\frac{x^5}{80p^2}\right)\Big|_0^{2p} = 4\pi p^3-\frac{8\pi p^3}{3}+\frac{4\pi p^3}{5}=\frac{32\pi p^3}{15}\rightarrow\frac{32}{15}$$

12.

$$\frac{\left(L^2-3^2-U^2\right)^2-4\left(3U\right)^2}{\left(L^2-U^2-6L+9\right)\left(L^2+3L+3U-U^2\right)}=\frac{\left(L^2-U^2-9-6U\right)\left(L^2-U^2-9+6U\right)}{\left(\left(L-3\right)^2-U^2\right)\left(\left(L-U\right)\left(L+U\right)+3\left(L+U\right)\right)}$$

$$\frac{\left[L^2-\left(U+3\right)^2\right]\left[L^2-\left(U-3\right)^2\right]}{\left(L-3-U\right)\left(L-3+U\right)\left(L+U\right)\left(L-U+3\right)}=\frac{\left(L+U+3\right)\left(L-3-U\right)\left(L-3+U\right)\left(L-U+3\right)}{\left(L-3-U\right)\left(L-3+U\right)\left(L+U\right)\left(L-U+3\right)}$$

$$\frac{L+U+3}{L+U}$$

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