2021 Nationals Alpha Bowl ANSWERS

1. 337
2. $\frac{10}{3}$
3. 68
4. $\frac{17}{9}$
5. 134
6. 8
7. 13124
8. 4042
9. $\frac{7\pi^2}{16}$
10. $2082 + 115\pi$
11. 8
12. $7 + 4\pi$
13. 5
14. $2 - \sqrt{3}$

2021 Nationals Alpha Bowl Question 1

Let $A = \sum_{n=10}^{4052} 2021$ Let $B = \sum_{n=1}^{2021} n$

Let $C = \sum_{n=1}^{2021} n^2$ Let $D = \sum_{n=1}^{2021} n^3$

Find $\frac{C}{A} + B^2 - D$

SOLUTION

$A = \sum_{n=10}^{4052} 2021 = 2021(4052 - 10 + 1) = (2021)(4043)$.

$B = \sum_{n=1}^{2021} n = \frac{2021(2022)}{2}$.

$C = \sum_{n=1}^{2021} n^2 = \frac{2021(2022)(4043)}{6}$.

$D = \sum_{n=1}^{2021} n^3 = \frac{2021^2 2022^2}{4}$.

Therefore $B^2 - D = 0$ and $\frac{C}{A} = \frac{2022}{6} = \boxed{337}$

2021 Nationals Alpha Bowl Question 2

Let $\vec{v} =< 1, 2, 2 >$ and let $\vec{w} =< 2, 3, 6 >$

Let X be the cosine of the acute angle between these vectors.

Let Y be the magnitude of the cross product between these vectors.

Let $Bx + Cy - z + D = 0$ be the equation of the plane containing these vectors and the point $(20,2,1)$

Let $\frac{x-E}{F} = y = \frac{z-G}{H}$ be the symmetric equations of the line in the direction of $\vec{w}$ containing the point $(2,0,21)$.

Find $7X + 2Y^2 + B + C + D + E + F + G + H$

SOLUTION

$X = \cos(\theta) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{20}{3*7} = \frac{20}{21}$.

$Y = \|\vec{v}\|\|\vec{w}\| \sin(\theta) = 21\sqrt{1 - \frac{20^2}{21^2}} = \sqrt{41}$.

The normal vector is

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 3 & 6 \end{vmatrix} =< 6, -2, -1 > \to 6(x - 20) - 2(y - 2) - (z - 1) = 6x - 2y - z - 115 = 0$$

The symmetric equations are $\frac{x-2}{2} = \frac{y-0}{3} = \frac{z-21}{6} \to \frac{x-2}{2/3} = y = \frac{z-21}{2}$

$$7X + 2Y^2 + B + C + D + E + F + G + H = \frac{20}{3} + 82 + 6 - 2 - 115 + 2 + \frac{2}{3} + 21 + 2 = \boxed{\frac{10}{3}}$$

2021 Nationals Alpha Bowl Question 3

Anagh wishes to form a committee with seven people. If he has three Thetas, five Alphas, and four Mus to choose uniformly at random from, let A be the probability that the committee has two Thetas, three Alphas, and two Mus.

The probability of Luke knowing how to solve Question #30 on a FAMAT test is $\frac{9}{10}$. If he doesn't know how to solve the question, he will guess one of the five choices with equal likelihood. If he does know how to solve the question, he will always get it correct. Given that Luke gets Question #30 correct, let B be the probability that Luke guessed.

Assume in a certain population that among all twin births, 60% of them are of the same gender. Assume that identical twins will always be the same gender, and that non-identical twins will be the same gender only 50% of the time. Let C be the proportion of twin births that are identical twins in this population.

Find $\frac{1}{AC}+\frac{1}{B}$.

SOLUTION

$A=\frac{\binom{3}{2}\binom{5}{3}\binom{4}{2}}{\binom{12}{7}}=\frac{3*10*6}{\frac{12*11*10*9*8}{5*4*3*2}}=\frac{5}{22}$.

$B=P(Guess|Correct)=\frac{P(Guess)P(Correct|Guess)}{P(Guess)P(Correct|Guess)+P(\sim Guess)P(Correct|\sim Guess)}=\frac{\left(\frac{1}{10}\right)\left(\frac{1}{5}\right)}{\left(\frac{1}{10}\right)\left(\frac{1}{5}\right)+\left(\frac{9}{10}\right)(1)}=\frac{1}{46}$.

$P(SameGender)=P(Identical)P(SameGender|Identical)+P(NotIdentical)P(SameGender|NotIdentical)\rightarrow 0.60=P(Identical)(1)+\big(1-P(Identical)\big)(0.50)\rightarrow 0.10=0.50*P(Identical)\rightarrow P(Identical)=\frac{1}{5}$.

$$\frac{1}{AC}+\frac{1}{B}=\frac{1}{\left(\frac{5}{22}\right)\left(\frac{1}{5}\right)}+\frac{1}{\frac{1}{46}}=22+46=\boxed{68}$$

2021 Nationals Alpha Bowl Question 4

Let $A = \lim_{x\to 1} \frac{x^2-2x+1}{x^2-1}$

Let $B = \lim_{x\to \infty} \frac{x^2-2x+1}{x^2-1}$

Let $C = \lim_{x\to 1} \frac{x^3+x^2-2}{3x^3-3}$

Let $D = \lim_{x\to \infty} \frac{x^3+x^2-2}{3x^3-3}$

Find $A + B + C + D$

SOLUTION

$A = \lim_{x\to 1} \frac{x^2-2x+1}{x^2-1} = \lim_{x\to 1} \frac{(x-1)^2}{(x-1)(x+1)} = 0.$

$B = \lim_{x\to \infty} \frac{x^2-2x+1}{x^2-1} = 1.$

$C = \lim_{x\to 1} \frac{x^3+x^2-2}{3x^3-3} = \lim_{x\to 1} \frac{(x-1)(x^2+2x+2)}{3(x-1)(x^2+x+1)} = \frac{5}{9}.$

$D = \lim_{x\to \infty} \frac{x^3+x^2-2}{3x^3-3} = \frac{1}{3}.$

$A + B + C + D = 0 + 1 + \frac{5}{9} + \frac{1}{3} = \boxed{\frac{17}{9}}.$

Alan and Srijan are running around a 21m circular track. Alan runs clockwise at 43 m/hr and Srijan runs counterclockwise at 47 m/hr. They start at the same spot on the track and run for 5 hours. Let A be the number of times they pass each other after they start running.

John only owns two types of books: comic books and math books. 1/6 of his books are comic books. After going to a book sale, he buys 10 more comic books, so 1/4 of his books are now comic books. Let B be the total number of books he has after the book sale.

Henry is trying to decide whether to take the stairs or the elevator. If he takes the stairs, it will take him 25 seconds to walk up each flight of stairs. If he takes the elevator, he will have to wait for 4 minutes for the elevator to arrive, after which it will take 5 seconds to move up each floor. Let C be the minimum number of floors for which taking the elevator takes less time than taking the stairs.

Find $A + B + C$

SOLUTION

Alan runs a total of 43*5=215 m and Srijan runs a total of 47*5=237 m. Since they are running in opposite directions, they will cross each other precisely when they have run a combined 21 m. The number of times this happens is $\left\lfloor \frac{215+237}{21} \right\rfloor = 21 = A$

Let be the number of comic books and be the number of math books. Then $\frac{x}{x+y} = \frac{1}{6} \to x = \frac{1}{5}y$.
$\frac{x+10}{x+y+10} = \frac{1}{4} \to \frac{\frac{3}{4}x+\frac{30}{4}}{\frac{1}{4}} = y = 3x + 30 \to x = \left(\frac{1}{5}\right)(3x+30) = \frac{3}{5}x + 6 \to x = \frac{6}{\frac{2}{5}} = 15 \to y = 75$. So after the book sale he owns $B = 15 + 75 + 10 = 100$ books

Say Henry wants to ascend n floors. By stairs, this take 25n seconds; by elevator, this takes 240 + 5n seconds. Solving the inequality 240 + 5n < 25n, we get 240/20 < n so 12 < n so the minimum is $C = 13$

$$A + B + C = 21 + 100 + 13 = \boxed{134}$$

Let A be the maximum possible value of the determinant of $\begin{bmatrix} 2x & 1-x & 0 \\ 2-x & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.

Consider the system of equations:

$$\begin{gathered} x+4y-2z+3w=2 \\ 2x-y-z+15w=10 \\ 3x+y-5z+6w=4 \\ 4x+2y+z+9w=6 \end{gathered}$$

Let B be the value of w in the solution to this system.

Let C and D be the eigenvalues of $\begin{bmatrix} 2 & 9 \\ 4 & 2 \end{bmatrix}$.

Find $AB+C+D$

SOLUTION

$\begin{vmatrix} 2x & 1-x & 0 \\ 2-x & 1 & 1 \\ 2 & 0 & 2 \end{vmatrix} = 4x+2(1-x)-2(2-x)(1-x) = -2x^2+8x-2$. The vertex of this parabola will be at $x=-\frac{b}{2a}=-\frac{8}{-4}=2$. So the maximum value is $-2(2)^2+8(2)-2=-8+16-2=6$

Using Cramer's rule, $w=\dfrac{\begin{vmatrix} 1 & 4 & -2 & 2 \\ 2 & -1 & -1 & 10 \\ 3 & 1 & -5 & 4 \\ 4 & 2 & 1 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -2 & 3 \\ 2 & -1 & -1 & 15 \\ 3 & 1 & -5 & 6 \\ 4 & 2 & 1 & 9 \end{vmatrix}} = \frac{2}{3}$ since the fourth column in the numerator is just 2/3 that of the denominator, and all other entries are the same.

$\begin{vmatrix} 2-y & 9 \\ 4 & 2-y \end{vmatrix} = (2-y)^2-36=0 \rightarrow 2-y=\pm 6 \rightarrow y=-4, 8.$

$$AB+C+D=6\left(\frac{2}{3}\right)+8-4=\boxed{8}$$

2021 Nationals Alpha Bowl Question 7

Let A be the number of distinct arrangements of the word *COMBINATORICS*

Let B be the number of positive integer solutions of the form (X_1, X_2, X_3) to the equation

$$X_1 + X_2 + X_3 = 2021$$

Let C be the number of terms in the expansion of $(X_1 + X_2 + X_3)^{2021}$

Dr. Santos has 16 (indistinguishable) black Math Competition shirts and 5 (indistinguishable) gold Math Competition shirts. Let D be the number of ways he can hang these shirts in his (linear) closet so that each gold shirt is separated by at least two black shirts. Gold shirts may be at the leftmost or rightmost end.

Find:

$$\frac{1}{5!}\frac{A}{D} + \frac{{}_{2023}P_5}{BC}$$

SOLUTION

$A = \frac{13!}{2!2!2!}$.

$B = \binom{2021-1}{3-1} = \frac{2020!}{2018!2!}$. One can see this by considering that if we represent our numbers by hashes of ones, 1…1+1….1+1….1 would represent a unique solution. So we just need to pick where to put our 3-1=2 plus signs from among the 2021-1=2020 spaces between the hashes.

$C = \binom{2021+3-1}{3-1} = \frac{2023!}{2021!2!}$. One can see this by noting that it is the same question as B, but with non-negative integers. So, we can that the 2021 ones and two pluses, and any arrangement of those would be a solution. For example, 11++11…11 would represent (2,0,2019).

For part D, we can represent this situation by $X_1GX_2GX_3GX_4GX_5GX_6$ where X_i represents the number of black shirts between gold shirt $i+1$ and gold shirt i, and $X_1, X_6 \geq 0$ but $X_2, X_3, X_4, X_5 \geq 2$. We need to find the number of solutions so that $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 16$. But this is hard because of the different bounds involved. So we can simplify things by letting $Y_1 = X_1 + 1, Y_6 = X_6 + 1, \& \, Y_i = X_i - 1$ for $2 \leq i \leq 5$. Now this is the solution to $(X_1 + 1) + (X_2 - 1) + (X_3 - 1) + (X_4 - 1) + (X_5 - 1) + (X_6 + 1) = 16 - (5 - 2 + 1) + 2 \to Y_1 + Y_2 + \cdots + Y_5 + Y_6 = 14$ for $Y_k > 0$. Thus this is the same as part B at this point and $D = \binom{14-1}{6-1} = \frac{(13)!}{(8)!(5)!}$

The final answer is

$$\frac{1}{5!}\frac{\frac{13!}{2!\,2!\,2!}}{\frac{13!}{8!\,5!}} + \frac{\frac{2023!}{2018!}}{\frac{2020!}{2018!\,2!}\frac{2023!}{2021!\,2!}} = 7! + 2021 * 4 = \boxed{13124}$$

2021 Nationals Alpha Bowl Question 8

Let $A = \sum_{n=0}^{\infty}\left(\frac{1}{2021^n}\right)$ Let $B = \sum_{n=1}^{\infty}\left(\frac{n}{2021^n}\right)$

Let $C = \sum_{n=2021}^{\infty}\left(\frac{1}{n^2+n}\right)$ Let $D = \sum_{n=2}^{2020}\left(\ln\left(\frac{n-1}{n+1}\right)\right)$

Find:

$$\frac{2Be^{-D}}{A} + \frac{1}{C}$$

SOLUTION

$A = \sum_{n=0}^{\infty}\left(\frac{1}{2021^n}\right) = \frac{1}{1-\frac{1}{2021}} = \frac{2021}{2020}.$

$B = \sum_{n=1}^{\infty}\left(\frac{n}{2021^n}\right) = \frac{\left(\frac{1}{2021}\right)}{\left(1-\frac{1}{2021}\right)^2} = \frac{\left(\frac{1}{2021}\right)}{\frac{2020^2}{2021^2}} = \frac{2021}{2020^2}.$

$C = \sum_{n=2021}^{\infty}\left(\frac{1}{n^2+n}\right) = \sum_{n=2021}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2021} - \frac{1}{2022} + \frac{1}{2022} - \frac{1}{2023} + \frac{1}{2023} - \cdots = \frac{1}{2021}.$

$D = \sum_{n=2}^{2020}\left(\ln\left(\frac{n-1}{n+1}\right)\right) = \ln\left(\frac{1}{3}\right) + \ln\left(\frac{2}{4}\right) + \ln\left(\frac{3}{5}\right) + \ln\left(\frac{4}{6}\right) + \cdots + \ln\left(\frac{2019}{2021}\right) = \ln\left(\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdot\ldots\cdot\frac{2017}{2019}\cdot\frac{2018}{2020}\cdot\frac{2019}{2021}\right) = \ln\left(\frac{2}{2020\cdot 2021}\right).$

$$\frac{Be^D}{A} + \frac{1}{C} = \frac{2\,\frac{2021}{2020^2}\cdot\frac{2020\cdot 2021}{2}}{\frac{2021}{2020}} + 2021 = \boxed{4042}$$

Let $A = \sin(2\alpha)$ if

$$\frac{\pi}{4} = \sin(\alpha + \sin(\alpha + \sin(\alpha + \cdots)))$$

Let B be the sum of all values of $0 < \theta < \pi$ for which

$$\frac{1+\sqrt{5}}{2} = \frac{\tan(\theta)}{1 - \frac{\tan(\theta)}{1 - \frac{\tan(\theta)}{1 - \cdots}}}$$

Let $C = \ln\left(e^{\frac{\pi i}{2} + 2\ln\left(e^{\frac{\pi i}{2} + 2\ln\left(e^{\frac{\pi i}{2} + 2\ln(\cdots)}\right)}\right)}\right)$ if the arguments of complex numbers are limited to $(-\pi, \pi]$.

Find $A + B^2 + C^2 + 1$

SOLUTION

$\frac{\pi}{4} = \sin(\alpha + sin(\alpha + sin(\alpha + \cdots))) = \sin\left(\alpha + \frac{\pi}{4}\right) = \sin(\alpha)\cos\left(\frac{\pi}{4}\right) + \cos(\alpha)\sin\left(\frac{\pi}{4}\right) \to \sin(\alpha) + \cos(\alpha) = \frac{\pi\sqrt{2}}{4} \to \sin^2(\alpha) + \cos^2(\alpha) + 2\sin(\alpha)\cos(\alpha) = \frac{\pi^2}{8} \to \sin(2\alpha) = \frac{\pi^2}{8} - 1 = A.$

Let $\varphi = \frac{1+\sqrt{5}}{2} = \frac{\tan(\theta)}{1 - \frac{\tan(\theta)}{1 - \frac{\tan(\theta)}{1 - \cdots}}} = \frac{\tan(\theta)}{1-\varphi} \to \varphi(1-\varphi) = \tan(\theta) = \frac{(1+\sqrt{5})(1-\sqrt{5})}{4} = -1 \to \theta = \frac{3\pi}{4} = B$

$C = \ln\left(e^{\frac{\pi i}{2} + 2\,ln\left(e^{\frac{\pi i}{2} + 2\,ln\left(e^{\frac{\pi i}{2} + 2\,ln(\cdots)}\right)}\right)}\right) \to C = \ln\left(e^{\frac{\pi i}{2} + 2C}\right) = \frac{\pi i}{2} + 2C \to C = -\frac{\pi i}{2}.$

$$A + B^2 + C^2 + 1 = \frac{\pi^2}{8} - 1 + \frac{9\pi^2}{16} - \frac{\pi^2}{4} + 1 = \boxed{\frac{7\pi^2}{16}}$$

2021 Nationals Alpha Bowl Question 10

Let A be the area contained within the graph of $\frac{(x-2021)^2}{25} + \frac{(y+2021)^2}{9} = 1$

Let B be the area contained within the graph of $|x| + |y| = 21$

Let C be the area contained within the polar graph of $r = 20\sin(\theta)$

Let D be the area of the rectangle with points on the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$ that has as two of its sides the latera recta of this ellipse.

Find $A + B + C + 13D$

SOLUTION

The area within an ellipse is $A = \pi ab = 15\pi$

The second curve is a square with semi-diagonal 21, so it has side length $21\sqrt{2}$ and therefore area $B = 882$

The third curve is a circle of radius 10, so it has area $C = 100\pi$

The distance from the center of the ellipse to a latus rectum is $c = \sqrt{a^2 - b^2} = 12$ The length of a latus rectum is $\frac{2b^2}{a} = \frac{50}{13}$. So the area is $D = (2 * 12)\left(\frac{50}{13}\right) = \frac{1200}{13}$.

$$A + B + C + 13D = \boxed{2082 + 115\pi}$$

Consider the conic section $rx^2 + (1-r)xy + y^2 + 2y - 2 = 0$ for real r.

Let A be the sum of all values of r for which this conic is a circle.

Let B be the sum of all values of r for which this conic is a parabola.

Let C be the sum of all values of r for which this conic is degenerate.

Using rotation of variables, this conic can be written in the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ in the rotated x'-y' plane. Let θ be any angle of rotation for which this will be true for all values of $r \neq 1$, and let $D = \sin^2(\theta)\cos^2(\theta)$.

Find $A + B + CD$

SOLUTION

Circles have no xy term, so the only way this could possibly be a circle is if $r = 1$. Then we have $x^2 + y^2 + 2y - 2 = 0 \rightarrow x^2 + (y+1)^2 = 3$, which is indeed a circle. So $A = 1$

$B^2 - 4AC = (1-r)^2 - 4r = r^2 - 6r + 1 = 0$. The sum of the roots of this quadratic is $B = 6$

Degeneracy is determined via the determinant $\begin{vmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{vmatrix} = \begin{vmatrix} r & \frac{1-r}{2} & 0 \\ \frac{1-r}{2} & 1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \frac{r^2}{2} - 4r + \frac{1}{2} = 0$

The sum of the roots of this quadratic is $C = 8$

An angle of rotation is found using $\cot(2\theta) = \frac{A-C}{B} = \frac{r-1}{1-r} = -1 \rightarrow \sin^2(\theta)\cos^2(\theta) = \frac{1}{4}\sin^2(2\theta) = \frac{1}{4\csc^2(2\theta)} = \frac{1}{4(1+\cot^2(2\theta))} = \frac{1}{8}$

$$A + B + CD = 1 + 6 + 8\left(\frac{1}{8}\right) = \boxed{8}$$

2021 Nationals Alpha Bowl Question 12

Let A be the sum of the real solutions to:

$$2\log_4(x) + \log_4(x^2 - 2x + 1) = 1$$

Let B be the number of distinct real solutions to:

$$(x^3 + 2x^2 - x - 1)^{(x^2+5x+4)} = 1$$

Let C be the sum of the real solutions for $0 < x \leq 2\pi$ to:

$$2\cos^3(x) - 5\cos^2(x) + \cos(x) + 2 = 0$$

Find $A + B + C$

SOLUTION

$2\log_4(x) + \log_4(x^2 - 2x + 1) = 1 \rightarrow 2\log_4(x) + 2\log_4(x-1) = 1 \rightarrow \log_4(x) + \log_4(x-1) = \frac{1}{2} \rightarrow \log_4\big(x(x-1)\big) = \frac{1}{2} \rightarrow x^2 - x = 2 \rightarrow x^2 - x - 2 = 0 \rightarrow x = -1,2$. However, $x = -1$ does not work in the original equation so $A = 2$.

$(x^3 + 2x^2 - x - 1)^{(x^2+5x+4)} = 1$ when

(1) $x^3 + 2x^2 - x - 1 = 1 \rightarrow x^3 + 2x^2 - x - 2 = 0 \rightarrow x^2(x+2) - (x+2) = 0 \rightarrow (x+2)(x+1)(x-1) = 0 \rightarrow x = -2, -1, 1$
(2) $x^2 + 5x + 4 = 0 \rightarrow (x+1)(x+4) = 0 \rightarrow x = -1, -4$
(3) $x^3 + 2x^2 - x - 1 = -1$ and $x^2 + 5x + 4$ is an even integer. $x^3 + 2x^2 - x - 1 = -1 \rightarrow x(x^2 + 2x - 1) = 0 \rightarrow x = 0, -1 \pm \sqrt{2}$. $x^2 + 5x + 4$ is an even integer only for $x = 0$.

There are $B = 5$ distinct solutions above.

$2\cos^3(x) - 5\cos^2(x) + \cos(x) + 2 = 0 \rightarrow (2\cos(x) + 1)(\cos(x) - 1)(\cos(x) - 2) = 0 \rightarrow \cos(x) = \frac{1}{2} \rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$ OR $\cos(x) = 1 \rightarrow x = 2\pi$. The sum of the solutions is $C = 4\pi$.

$$A + B + C = \boxed{7 + 4\pi}$$

Let X be the binary representation of a number such that

$$21_{20} + 20_{21} + 20_{2021} + 2021_3 = X_2$$

How many times does the digit 1 appear in X?

SOLUTION

$21_{20} = 2 * 20 + 1 = 41.$

$20_{21} = 2 * 21 = 42.$

$20_{2021} = 2 * 2021 = 4042.$

$2021_3 = 2 * 3^3 + 2 * 3 + 1 = 61.$

The total is 4186, which in binary is 1000001011010_2, so the answer is $\boxed{5}$

2021 Nationals Alpha Bowl Question 14

Evaluate:

$$\frac{2\sqrt{\sec^2\left(\frac{\pi}{24}\right)-1}}{1-\frac{1-\cos\left(\frac{\pi}{12}\right)}{1+\cos\left(\frac{\pi}{12}\right)}}$$

SOLUTION

$$\frac{2\sqrt{\sec^2\left(\frac{\pi}{24}\right)-1}}{1-\frac{1-\cos\left(\frac{\pi}{12}\right)}{1+\cos\left(\frac{\pi}{12}\right)}} = \frac{2\tan\left(\frac{\pi}{24}\right)}{1-\frac{2\sin^2\left(\frac{\pi}{24}\right)}{2\cos^2\left(\frac{\pi}{24}\right)}} = \frac{2\tan\left(\frac{\pi}{24}\right)}{1-\tan^2\left(\frac{\pi}{24}\right)} = \tan\left(\frac{\pi}{12}\right) = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{\left(\sqrt{6}-\sqrt{2}\right)^2}{6-2} = \frac{6+2-2\sqrt{12}}{4} = \boxed{2-\sqrt{3}}.$$