2021 Nationals Mu Bowl ANSWERS

1. 3982
2. 13
3. 68
4. 12
5. $3031 + \ln(2)$
6. 21
7. 13124
8. $e^3 - 2$
9. 75
10. 94
11. 8
12. $\frac{24\ln(2)}{31}$
13. $\frac{1}{252}$
14. $9 + 6\pi$

2021 Nationals Mu Bowl Question 1

Let $f(x) = 2021 - 12x^3 - 6x$. Let R be the finite region bounded by $f(x)$, the x-axis, $x = 0$, and $x = 2$.

Let A be the value obtained when the area of R is approximated using a Left-handed Riemann Sum with 8 equal subintervals.

Let B be the value obtained when the area of R is approximated using a Right-handed Riemann Sum with 8 equal subintervals.

Let C be the value obtained when the area of R is approximated using the Trapezoidal Rule with 8 equal subintervals.

Let D be the value obtained when the area of R is approximated using Simpson's Rule with 8 equal subintervals.

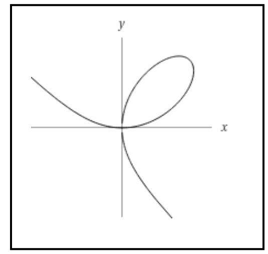
Find $A + B - 2C + D$

SOLUTION

$\frac{A+B}{2} = C$ so $A + B - 2C = 0$.

Simpson's Rule is exact for cubics (and below) so $D = \int_0^2 (2021 - 12x^3 - 6x)dx = [-3x^4 - 3x^2 + 2021x]_0^2 = -48 - 12 + 4042 = \boxed{3982}$.

2021 Nationals Mu Bowl Question 2



Consider the Folium of Descartes pictured at right with the equation

$$x^3 + y^3 = 3axy; \quad a > 0$$

Let (A, B) be the coordinates of the point in the first quadrant that is furthest from the origin.

Let (C, D) be the coordinates of the point with a horizontal tangent line that has the largest y-value.

Let (E, F) be the coordinates of the point with a vertical tangent line that has the largest y-value.

Let $y = Hx + G$ be the equation of the oblique asymptote of this curve.

Find

$$\frac{A}{a} + \frac{B}{a} + \left(\frac{C}{a}\right)^3 + \left(\frac{D}{a}\right)^3 + \left(\frac{E}{a}\right)^3 + \left(\frac{F}{a}\right)^3 + \frac{G}{a} + H$$

SOLUTION

In general, we will use the fact that the curve is symmetric about $y = x$, and also that the derivative is found using implicit differentiation: $3x^2 + 3y^2y' = 3ay + 3axy' \rightarrow \frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax}$

The point furthest from the origin is just the intersection with $y = x$, so $2x^3 = 3ax^2 \rightarrow x = \frac{3a}{2}$. So the coordinate is $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

The curve has a horizontal tangent line when $3ay - 3x^2 = 0 \rightarrow y = \frac{x^2}{a}$. Plugging this in yields $x^3 + \frac{x^6}{a^3} = 3x^3 \rightarrow x^6 = 2x^3a^3 \rightarrow x = a\sqrt[3]{2} \rightarrow y = a\sqrt[3]{4} \rightarrow (C, D) = \left(a\sqrt[3]{2}, a\sqrt[3]{4}\right)$

By symmetry, $(E, F) = \left(a\sqrt[3]{4}, a\sqrt[3]{2}\right)$

As $x \rightarrow \infty, y \approx Hx + G$ so $1 = \lim_{x\rightarrow\infty} \frac{x^3 + (Hx+G)^3}{3ax(Hx+G)} = \lim_{x\rightarrow\infty} \frac{(1+H^3)x^3 + 3H^2Gx^2 + 3HG^2x + G^3}{3aHx^2 + 3Gax}$. This limit will only exist and be one if $1 + H^3 = 0$ and $3H^2G = 3aH$. So $H = -1$ and $G = -a$

$$\frac{A}{a} + \frac{B}{a} + \left(\frac{C}{a}\right)^3 + \left(\frac{D}{a}\right)^3 + \left(\frac{E}{a}\right)^3 + \left(\frac{F}{a}\right)^3 + \frac{G}{a} + H = \frac{3}{2} + \frac{3}{2} + 2 + 4 + 4 + 2 - 1 - 1 = \boxed{13}$$

2021 Nationals Mu Bowl Question 3

Alex wishes to form a committee with seven people. If he has three Thetas, five Alphas, and four Mus to choose uniformly at random from, let A be the probability that the committee has two Thetas, three Alphas, and two Mus.

The probability of Saathvik knowing how to solve Question #30 on a FAMAT test is $\frac{9}{10}$. If he doesn't know how to solve the question, he will guess one of the five choices with equal likelihood. If he does know how to solve the question, he will always get it correct. Given that Saathvik gets Question #30 correct, let B be the probability that Saathvik guessed.

Assume in a certain population that among all twin births, 60% of them are of the same gender. Assume that identical twins will always be the same gender, and that non-identical twins will be the same gender only 50% of the time. Let C be the proportion of twin births that are identical twins in this population.

Find $\frac{1}{AC}+\frac{1}{B}$.

SOLUTION

$A=\frac{\binom{3}{2}\binom{5}{3}\binom{4}{2}}{\binom{12}{7}}=\frac{3*10*6}{\frac{12*11*10*9*8}{5*4*3*2}}=\frac{5}{22}$.

$B=P(Guess|Correct)=\frac{P(Guess)P(Correct|Guess)}{P(Guess)P(Correct|Guess)+P(\sim Guess)P(Correct|\sim Guess)}=\frac{\left(\frac{1}{10}\right)\left(\frac{1}{5}\right)}{\left(\frac{1}{10}\right)\left(\frac{1}{5}\right)+\left(\frac{9}{10}\right)(1)}=\frac{1}{46}$.

$P(SameGender)=P(Identical)P(SameGender|Identical)+P(NotIdentical)P(SameGender|NotIdentical)\rightarrow 0.60=P(Identical)(1)+\big(1-P(Identical)\big)(0.50)\rightarrow 0.10=0.50*P(Identical)\rightarrow P(Identical)=\frac{1}{5}$.

$$\frac{1}{AC}+\frac{1}{B}=\frac{1}{\left(\frac{5}{22}\right)\left(\frac{1}{5}\right)}+\frac{1}{\frac{1}{46}}=22+46=\boxed{68}$$

2021 Nationals Mu Bowl Question 4

Let $A = \frac{d}{dx}\left[\sqrt{25 - x^2}\right]\Big]_{x=3}$

Let $B = \frac{d}{dx}[(x+1)(x^2+1)(x^3+1)]\Big]_{x=1}$

Let $C = \frac{d}{dx}[\ln(\sec(x) + \tan(x))]\Big]_{x=\frac{\pi}{4}}$

Let $D = \frac{d}{dx}[\arctan(cos(x))]\Big]_{x=\frac{\pi}{4}}$

Find $A \cdot B \cdot C \cdot D$

SOLUTION

$A = \frac{d}{dx}\left[\sqrt{25 - x^2}\right]\Big]_{x=3} = -\frac{x}{\sqrt{25-x^2}}\Big]_{x=3} = -\frac{3}{4}.$

$B = \frac{d}{dx}[(x+1)(x^2+1)(x^3+1)]\Big]_{x=1} = (x^2+1)(x^3+1) + 2x(x+1)(x^3+1) + 3x^2(x+1)(x^2+1)]_{x=1} = 4 + 8 + 12 = 24.$

$C = \frac{d}{dx}[\ln(\sec(x) + \tan(x))]\Big]_{x=\frac{\pi}{4}} = \sec(x)]_{x=\frac{\pi}{4}} = \sqrt{2}.$

$D = \frac{d}{dx}[\arctan(cos(x))]\Big]_{x=\frac{\pi}{4}} = -\frac{\sin(x)}{1+\cos^2(x)}\Big]_{x=\frac{\pi}{4}} = -\frac{\frac{\sqrt{2}}{2}}{1+\frac{1}{2}} = -\frac{\sqrt{2}}{3}.$

$$A \cdot B \cdot C \cdot D = \left(-\frac{3}{4}\right)(24)\left(\sqrt{2}\right)\left(-\frac{\sqrt{2}}{3}\right) = \boxed{12}$$

2021 Nationals Mu Bowl Question 5

Let $A = \frac{3}{2}\int_0^{2020}(x-1010)^5 + (x-1010)^3 + (x-1010)^2\,dx$

Let $B = \int_0^{2021}\frac{x^{2021}}{x^{2021}+(2021-x)^{2021}}\,dx$

Let $C = \int_0^{\pi}\frac{x\sin(x)}{1+\cos^2(x)}\,dx$

Let $D = \int_0^{\infty}\frac{2\ln(x)}{x^2+4}\,dx$

Find

$$\sqrt[3]{A} + 2B + \frac{D}{\sqrt{C}}$$

SOLUTION

$A = \frac{3}{2}\int_0^{2020}(x-1010)^5 + (x-1010)^3 + (x-1010)^2\,dx$. Let $u = x - 1010$. Then $A = \frac{3}{2}\int_{-1010}^{1010}u^5 + u^3 + u^2\,du = 3\int_0^{1010}u^2\,du = (1010)^3$

$B = \int_0^{2021}\frac{x^{2021}}{x^{2021}+(2021-x)^{2021}}\,dx$. Let $u = 2021 - x$. Then $B = -\int_{2021}^{0}\frac{(2021-u)^{2021}}{(2021-u)^{2021}+u^{2021}}\,du = \int_0^{2021}\frac{(2021-u)^{2021}}{(2021-u)^{2021}+u^{2021}}\,du$. Then $2B = B + B = \int_0^{2021}\frac{x^{2021}}{x^{2021}+(2021-x)^{2021}}\,dx + \int_0^{2021}\frac{(2021-x)^{2021}}{(2021-x)^{2021}+x^{2021}}\,dx = \int_0^{2021}1\,dx = 2021$. So $B = \frac{2021}{2}$

$C = \int_0^{\pi}\frac{x\sin(x)}{1+\cos^2(x)}\,dx$. Let $u = \pi - x$. Then $C = \int_0^{\pi}\frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}\,du = \int_0^{\pi}\frac{\pi\sin(u)}{1+\cos^2(u)}\,du - \int_0^{\pi}\frac{u\sin(u)}{1+\cos^2(u)}\,du = \int_0^{\pi}\frac{\pi\sin(u)}{1+\cos^2(u)}\,du - C$. Therefore $C = \frac{1}{2}\int_0^{\pi}\frac{\pi\sin(u)}{1+\cos^2(u)}\,du = \frac{\pi}{2}\int_{-1}^{1}\frac{1}{1+v^2}\,dv = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$

$D = \int_0^{\infty}\frac{2\ln(x)}{x^2+4}\,dx$. Let $u = \frac{4}{x} \rightarrow dx = -\frac{4}{u^2}du$. $D = \int_0^{\infty}\frac{2\ln\left(\frac{2}{u}\right)}{\frac{16}{u^2}+4}\frac{4}{u^2}\,du = \int_0^{\infty}\frac{2(\ln(4)-\ln(u))}{u^2+4}\,du = \int_0^{\infty}\frac{2\ln(4)}{u^2+4}\,du - D$. So $D = \ln(4)\int_0^{\infty}\frac{1}{u^2+4}\,du = \frac{\ln(4)\pi}{4} = \frac{\pi\ln(2)}{2}$

$$\sqrt[3]{A} + 2B + \frac{D}{\sqrt{C}} = 1010 + 2021 + \ln(2) = \boxed{3031 + \ln(2)}$$

2021 Nationals Mu Bowl Question 6

$f(x)$ and $g(x)$ are continuous, twice-differentiable functions. Using only the information in the table below, find the indicated values.

$x =$	1	2	3	4
$f(x)$	2	3	4	9
$f'(x)$	3	1	5	12
$g(x)$	-1	4	3	10
$g'(x)$	5	5	-1	2

Let $A = h'(1)$ if $h(x) = \frac{g(x)}{f(x)+1}$

Let $B = k'(2)$ if $k(x) = \sqrt{f\big(g(x)\big)}$

Let C be the average rate of change of $p(x) = f(x)g(x)$ on the interval $[1,3]$

Let D be the value of c guaranteed by the Mean Value Theorem for Derivatives on the interval $[1,4]$ for the function $f(x) + g(x)$

Find $A + B + C + D$

SOLUTION

$h'(x) = \frac{(f(x)+1)g'(x) - g(x)f'(x)}{(f(x)+1)^2} \to h'(1) = \frac{(2+1)(5)-(-1)(3)}{(2+1)^2} = \frac{18}{9} = 2 = A.$

$k'(3) = \frac{f'\big(g(2)\big)g'(2)}{2\sqrt{f\big(g(2)\big)}} = \frac{12*5}{2\sqrt{9}} = 10 = B.$

The average rate of change on that interval is $\frac{f(3)g(3)-f(1)g(1)}{3-1} = \frac{(4)(3)-(2)(-1)}{2} = 7 = C.$

We are looking for a value of $x = c$ for which $f'(c) + g'(c) = \frac{f(4)+g(4)-f(1)-g(1)}{4-1} = \frac{9+10-2+1}{3} = 6$. This occurs for $x = 2 = D$.

$$A + B + C + D = 2 + 10 + 7 + 2 = \boxed{21}$$

2021 Nationals Mu Bowl Question 7

Let A be the number of distinct arrangements of the word *COMBINATORICS*

Let B be the number of positive integer solutions of the form (X_1, X_2, X_3) to the equation

$$X_1 + X_2 + X_3 = 2021$$

Let C be the number of terms in the expansion of $(X_1 + X_2 + X_3)^{2021}$

Jae has 16 (indistinguishable) black Math Competition shirts and 5 (indistinguishable) gold Math Competition shirts. Let D be the number of ways he can hang these shirts in his (linear) closet so that each gold shirt is separated by at least two black shirts. Gold shirts may be at the leftmost or rightmost end.

Find:

$$\frac{1}{5!}\frac{A}{D} + \frac{{}_{2023}P_5}{BC}$$

SOLUTION

$A = \frac{13!}{2!2!2!}$.

$B = \binom{2021-1}{3-1} = \frac{2020!}{2018!2!}$. One can see this by considering that if we represent our numbers by hashes of ones, 1…1+1….1+1….1 would represent a unique solution. So we just need to pick where to put our 3-1=2 plus signs from among the 2021-1=2020 spaces between the hashes.

$C = \binom{2021+3-1}{3-1} = \frac{2023!}{2021!2!}$. One can see this by noting that it is the same question as B, but with non-negative integers. So, we can that the 2021 ones and two pluses, and any arrangement of those would be a solution. For example, 11++11…11 would represent (2,0,2019).

For part D, we can represent this situation by $X_1GX_2GX_3GX_4GX_5GX_6$ where X_i represents the number of black shirts between gold shirt $i+1$ and gold shirt i, and $X_1, X_6 \geq 0$ but $X_2, X_3, X_4, X_5 \geq 2$. We need to find the number of solutions so that $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 16$. But this is hard because of the different bounds involved. So we can simplify things by letting $Y_1 = X_1 + 1, Y_6 = X_6 + 1, \& \, Y_i = X_i - 1$ for $2 \leq i \leq 5$. Now this is the solution to $(X_1+1) + (X_2-1) + (X_3-1) + (X_4-1) + (X_5-1) + (X_6+1) = 16 - (5-2+1) + 2 \to Y_1 + Y_2 + \cdots + Y_5 + Y_6 = 14$ for $Y_k > 0$. Thus this is the same as part B at this point and $D = \binom{14-1}{6-1} = \frac{(13)!}{(8)!(5)!}$

The final answer is

$$\frac{1}{5!}\frac{\frac{13!}{2!\,2!\,2!}}{\frac{13!}{8!\,5!}} + \frac{\frac{2023!}{2018!}}{\frac{2020!}{2018!\,2!}\frac{2023!}{2021!\,2!}} = 7! + 2021 * 4 = \boxed{13124}$$

2021 Nationals Mu Bowl Question 8

Let $A = y(1)$ if $y' = 2xy + 2x + y + 1$ and $y(0) = 0$

Let $B = y(1)$ if $y' + xy = 6e^{3x}$ and $y(0) = 0$

Let $C = y(1)$ if $(e^{3x} - 1)y' + 3e^{3x}y = 0$ and $\lim_{x \to -\infty} y = 1$

Let $D = y(-1)$ if $(e^{3x} - 1)y'' + 6e^{3x}y' + 9e^{3x}y = 0$ and $\lim_{x \to -\infty} y = 2$

Find

$$(A + 1)e + B + \frac{D}{C}$$

SOLUTION

$y' = 2xy + 2x + y + 1 = (2x + 1)(y + 1) \to \frac{1}{y+1}y' = 2x + 1 \to \ln(y + 1) = x^2 + x + C \to y = Ce^{x^2+x} - 1.$ $y(0) = 0 \to C = 1 \to y = e^{x^2+x} - 1 \to y(1) = e^2 - 1 = A$

$y' + xy = 6e^{3x} = [xy]' \to xy = 2e^{3x} + C.$ $y(0) = 0 \to 0 = 2 + C \to C = -2 \to xy = 2e^{3x} - 2 \to (1)y(1) = y(1) = 2e^3 - 2 = B$

$(e^{3x} - 1)y' + 3e^{3x}y = 0 \to y' = e^{3x}y' + 3e^{3x}y = [e^{3x}y]' \to y = e^{3x}y + Q \to y = \frac{Q}{1-e^{3x}}.$ $\lim_{x \to -\infty} y = 1 = \frac{Q}{1} \to Q = 1 \to y = \frac{1}{1-e^{3x}} \to y(1) = \frac{1}{1-e^3} = C$

$(e^{3x} - 1)y'' + 6e^{3x}y' + 9e^{3x}y = 0 \to y'' = e^{3x}y'' + 6e^{3x}y' + 9e^{3x}y = [e^{3x}y]'' \to y' = [e^{3x}y]' + C \to y = e^{3x}y + Cx + D \to y = \frac{Cx+D}{1-e^{3x}}.$ $\lim_{x \to -\infty} y = 2$ only when $C = 0$ and $D = 2.$ So $y = \frac{2}{1-e^{3x}} \to y(-1) = \frac{2}{1-e^{-3}} = \frac{2e^3}{e^3-1} = D$

$$(A + 1)e + B + \frac{D}{C} = e^3 + 2e^3 - 2 + \frac{\frac{2e^3}{e^3 - 1}}{\frac{1}{1 - e^3}} = \boxed{e^3 - 2}$$

2021 Nationals Mu Bowl Question 9

Let $A = \int_0^{\pi/6} x^3 \tan(x^5 + x)\, dx$ Let $B = \int_0^{\pi/6} (2x\sec(x) + \sec^2(x))dx$

Let $C = \int_0^{\pi/6} x^2 \sec(x) \tan\,(x)dx$ Let $D = \int_0^{\pi/6} \frac{x^3\tan(x)+x^3\tan(x^5)}{\tan(x^5)\tan(x)-1}dx$

If $A + B + C + D = \frac{(\pi^2+p)\sqrt{q}}{r}$, and q is square-free, find $p + q + r$

SOLUTION

$A = \int_0^{\pi/6} x^3 \tan(x^5 + x)\, dx = \int_0^{\pi/6} \frac{x^3\tan(x)+x^3\tan(x^5)}{1-\tan(x^5)\tan(x)}dx = -\int_0^{\pi/6} \frac{x^3\tan(x)+x^3\tan(x^5)}{\tan(x^5)\tan(x)-1}dx = -D$ using the tangent addition formula, so $A + D = 0$.

$B + C = \int_0^{\pi/6} 2x\sec(x) + \sec^2(x) + x^2\sec(x)\tan\,(x)dx = \int_0^{\pi/6}(2x\sec(x) + x^2\sec(x)\tan(x)) + \sec^2(x)\, dx = \int_0^{\pi/6}(x^2\sec(x))' + \sec^2(x)\, dx = \int_0^{\pi/6}(x^2\sec(x))' + (\tan(x))'dx = [x^2\sec(x) + \tan(x)]_0^{\pi/6} = \frac{\pi^2}{36}\left(\frac{2}{\sqrt{3}}\right) + \left(\frac{\sqrt{3}}{3}\right) = \frac{(\pi^2+18)\sqrt{3}}{54} \to 18 + 3 + 54 = \boxed{75}$

2021 Nationals Mu Bowl Question 10

For each of the following series, assign a value of 21 if it converges absolutely, 20 if it converges conditionally, and -10 if it diverges. Your final answer is the sum of assigned values.

$\sum_{n=1}^{\infty} \frac{(-1)^n n^{2021}}{2021^n}$	$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \frac{(2n)!}{(n!)^2}$	$\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$
$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$	$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$	$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$
$\sum_{n=1}^{\infty} 2021^{-n-(-1)^n}$	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

SOLUTION

Many of these series can be proven convergent or divergent multiple ways. Here we present the most direct approach:

Absolutely Convergent by the ratio test since $\lim_{n\to\infty} \frac{\frac{(n+1)^{2021}}{2021^{n+1}}}{\frac{n^{2021}}{2021^n}} = \frac{1}{2021} < 1$	Diverges by Raabe's test since $\lim_{n\to\infty} \left(n\left(\frac{\left(\frac{1}{4}\right)^n \frac{(2n)!}{(n!)^2}}{\left(\frac{1}{4}\right)^{n+1} \frac{(2n+2)!}{((n+1)!)^2}} - 1\right)\right) =$ $\lim_{n\to\infty} \left(n\left(\frac{4(n+1)^2}{(2n+2)(2n+1)} - 1\right)\right) =$ $\lim_{n\to\infty} \left(\frac{2n^2+2n}{(2n+2)(2n+1)}\right) = \frac{1}{2} < 1$	Conditionally convergent since $\sin(n)$ will behave similarly to $(-1)^n$ and this therefore behaves like the alternating harmonic series
Divergent since $\lim_{n\to\infty} \frac{n^2}{n^2+1} \neq 0$	Conditionally convergent since $\frac{1}{n \ln(n)} \to 0$ but $\int_2^{\infty} \frac{dx}{x \ln(x)} = \infty$	Absolutely Convergent by the ratio test or by Taylor's theorem for e^x
Absolutely Convergent by the root test since $\lim_{n\to\infty} \left(2021^{-n-(-1)^n}\right)^{\frac{1}{n}} =$ $\lim_{n\to\infty} \left(2021^{-1-\frac{(-1)^n}{n}}\right)^{\frac{1}{n}} =$ $\frac{1}{2021} < 1$	Divergent by the p-test.	By Sterling's approximation this will behave like $\sum_{n=1}^{\infty} (-1)^n \sqrt{2\pi n} e^{-n}$ which converges via the ratio test. So this is absolutely convergent

The total is $21 - 10 + 20 - 10 + 20 + 21 + 21 - 10 + 21 = \boxed{94}$

2021 Nationals Mu Bowl Question 11

Consider the conic section $rx^2 + (1-r)xy + y^2 + 2y - 2 = 0$ for real r.

Let A be the sum of all values of r for which this conic is a circle.

Let B be the sum of all values of r for which this conic is a parabola.

Let C be the sum of all values of r for which this conic is degenerate.

Using rotation of variables, this conic can be written in the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ in the rotated x'-y' plane. Let θ be any angle of rotation for which this will be true for all values of $r \neq 1$, and let $D = \sin^2(\theta)\cos^2(\theta)$.

Find $A + B + CD$

SOLUTION

Circles have no xy term, so the only way this could possibly be a circle is if $r = 1$. Then we have $x^2 + y^2 + 2y - 2 = 0 \rightarrow x^2 + (y+1)^2 = 3$, which is indeed a circle. So $A = 1$

$B^2 - 4AC = (1-r)^2 - 4r = r^2 - 6r + 1 = 0$. The sum of the roots of this quadratic is $B = 6$

Degeneracy is determined via the determinant $\begin{vmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{vmatrix} = \begin{vmatrix} r & \frac{1-r}{2} & 0 \\ \frac{1-r}{2} & 1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \frac{r^2}{2} - 4r + \frac{1}{2} = 0$

The sum of the roots of this quadratic is $C = 8$

An angle of rotation is found using $\cot(2\theta) = \frac{A-C}{B} = \frac{r-1}{1-r} = -1 \rightarrow \sin^2(\theta)\cos^2(\theta) = \frac{1}{4}\sin^2(2\theta) = \frac{1}{4\csc^2(2\theta)} = \frac{1}{4(1+\cot^2(2\theta))} = \frac{1}{8}$

$$A + B + CD = 1 + 6 + 8\left(\frac{1}{8}\right) = \boxed{8}$$

2021 Nationals Mu Bowl Question 12

For $f(x) = \frac{1}{x^2}$, consider the region between $f(x)$ and the x-axis from $x = 1$ to $x = 2$.

Let A be the area of this region.

Let B be the volume of this region when it is revolved around the line $y = -1$.

Let C be the volume of this region when it is revolved around the y-axis.

Find $\frac{A \cdot C}{B}$

SOLUTION

$A = \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$

$B = \pi \int_1^2 \left(\frac{1}{x^2} + 1\right)^2 - 1^2 \, dx = \pi \int_1^2 \frac{1}{x^4} + \frac{2}{x^2} \, dx = \pi \left[-\frac{\frac{1}{3}}{x^3} - \frac{2}{x}\right]_1^2 = \pi \left(-\frac{1}{24} - 1 + \frac{1}{3} + 2\right) =$
$\pi \left(\frac{7}{24} + 1\right) = \frac{31\pi}{24}.$

$C = 2\pi \int_1^2 x \left(\frac{1}{x^2}\right) \, dx = 2\pi [\ln(x)]_1^2 = 2\pi \ln(2).$

$$\frac{A \cdot C}{B} = \frac{24}{31\pi} (2\pi \ln(2)) \left(\frac{1}{2}\right) = \boxed{\frac{24 \ln(2)}{31}}$$

2021 Nationals Mu Bowl Question 13

Let A be the maximum value of $2x + 3y + 6z$ given that $x^2 + y^2 + z^2 = 16$.

Let B be the minimum possible value of $x^2 + y^2 + z^2$ given that $x + 4y + 8z = 27$.

Let C be the minimum possible value of $(x + y + z)^3$ given that $xyz = 8$ and $x, y, z > 0$.

Find the maximum possible value of xyz given that $Ax + By + Cz = 18$ and $x, y, z > 0$. Express your answer as a common fraction.

SOLUTION

$\boldsymbol{u} \cdot \boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\| \rightarrow 2x + 3y + 6z = < x, y, z > \cdot < 2,3,6 > \leq \sqrt{x^2 + y^2 + z^2}\sqrt{2^2 + 3^2 + 6^2} = 4 * 7 = 28.$

$\boldsymbol{u} \cdot \boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\| \rightarrow \sqrt{x^2 + y^2 + z^2}\sqrt{1^2 + 4^2 + 8^2} \geq < x, y, z > \cdot < 1,2,2 > = x + 2y + 2z = 27 \rightarrow \sqrt{x^2 + y^2 + z^2} \geq 3 \rightarrow x^2 + y^2 + z^2 \geq 9.$

$(xyz)^{1/3} \leq \frac{x+y+z}{3} \rightarrow x + y + z \geq 3 * 2 \rightarrow (x + y + z)^3 \geq 27 * 8 = 216.$

$(xyz)^{1/3} \leq \frac{x+y+z}{3} \rightarrow AxByCz \leq \frac{(Ax+By+Cz)^3}{27} \rightarrow xyz \leq \frac{18^3}{27*28*9*216} = \frac{1}{28*9} = \boxed{\frac{1}{252}}.$

2021 Nationals Mu Bowl Question 14

The function $f(x) = \begin{cases} \arctan(x) & x \geq 1 \\ Ax^3 + Bx^2 + Cx + D & x < 1 \end{cases}$ is continuous, as are $f'(x)$, $f''(x)$, and $f'''(x)$.

Find $24 \cdot (A \cdot B + C + D)$.

SOLUTION

For $x \geq 1$, $f(x) = \arctan(x) \rightarrow f'(x) = \frac{1}{1+x^2} \rightarrow f''(x) = -\frac{2x}{(1+x^2)^2} \rightarrow f'''(x) = \frac{(1+x^2)^2(-2)+8x^3(1+x^2)}{(1+x^2)^4}$

So $f(1) = \frac{\pi}{4}$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{2}$, and $f'''(x) = \frac{4(-2)+16}{16} = \frac{1}{2}$.

For $x < 1$, $f(x) = Ax^3 + Bx^2 + Cx + D \rightarrow f'(x) = 3Ax^2 + 2Bx + C \rightarrow f''(x) = 6Ax + 2B \rightarrow$ $f'''(x) = 6A$. Working backwards, $6A = \frac{1}{2} \rightarrow A = \frac{1}{12} \rightarrow -\frac{1}{2} = \frac{1}{2}(1) + 2B \rightarrow B = -\frac{1}{2} \rightarrow \frac{1}{2} = \frac{1}{4}(1)^2 - (1) + C \rightarrow C = \frac{5}{4} \rightarrow \frac{\pi}{4} = \frac{1}{12} - \frac{1}{2} + \frac{5}{4} + D \rightarrow D = \frac{\pi}{4} - \frac{5}{6}$.

$24(A \cdot B + C + D) = 24\left(\frac{1}{12}\left(-\frac{1}{2}\right) + \frac{5}{4} + \frac{\pi}{4} - \frac{5}{6}\right) = \boxed{9 + 6\pi}$.