1. A
2. D
3. B
4. B
5. A
6. B
7. D
8. A
9. C
10. C
11. C
12. B
13. D
14. A
15. D
16. E
17. D
18. C
19. A
20. B
21. C
22. E
23. C
24. A
25. C
26. B
27. D
28. D
29. A
30. B

1. **A** Our combination of three days consists of Friday and two days that are not Friday or Wednesday. There are 5 such days (Monday, Tuesday, Thursday, Saturday, and Sunday), so we have $\binom{5}{2} = \boxed{10}$ ways to choose the other two days.

2. **D** Let v denote the common velocity of the waves. The wavelength of A is given by $\lambda_A = \frac{v}{f} = \frac{v}{440}$ and the wavelength of E is given by $\lambda_E = \frac{v}{660}$. Taking the ratio, we get
$$\frac{\lambda_A}{\lambda_E} = \frac{\frac{v}{440}}{\frac{v}{660}} = \frac{660}{440} = \boxed{\frac{3}{2}}.$$

3. **B** Let $t = 0$ denote the instant Hina starts moving. The distances traveled by Hodaka and Hina can be modeled by $d = 15(t + 2)$ and $d = 20t$ respectively. Setting these equal, we get
$$15(t + 2) = 20t \Longrightarrow 5t = 30 \Longrightarrow t = \boxed{6}.$$

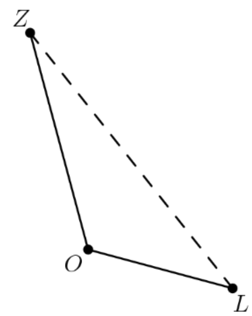
4. **B** The box moves in the direction of $\binom{1}{2}$, so we can call its final position $(d, 2d)$ for some real number d. The work done by Dietfried is given by $\begin{pmatrix} F \\ 0 \end{pmatrix} \cdot \begin{pmatrix} d \\ 2d \end{pmatrix} = Fd$, and the work done by Gilbert is given by $\begin{pmatrix} 0 \\ 2F \end{pmatrix} \cdot \begin{pmatrix} d \\ 2d \end{pmatrix} = 4Fd$. The desired ratio is thus $Fd : 4Fd = \boxed{1 : 4}$.

5. **A** 2 revolutions per second is equivalent to $2(2\pi) = 4\pi$ radians per second. The radius of the bike wheels is $\frac{1}{\pi}$ meters, so this corresponds to a linear speed of $4\pi\left(\frac{1}{\pi}\right) = 4$ meters per second. This means Mitsuha can reach Taki in $\frac{1000}{4} = 250$ seconds, which is $\frac{250}{60} \approx \boxed{4}$ minutes.

6. **B** Haku needs to take $\frac{100}{4} = 25$ sets of 4 steps, with 3 second pauses between each set. This corresponds to 24 pauses, so the total amount of time it takes is $25(4) + 24(3) = \boxed{172}$.

7. **D** The number of tasks Retsuko must complete on day k is $11 + 2(k - 1) = 2k + 9$. We want the smallest n such that $\sum_{k=1}^{n}(2k + 9) \geq 2022$. Since $\sum_{k=1}^{n} k = \frac{n^2+n}{2}$, we have that
$$\sum_{k=1}^{n}(2k + 9) = n^2 + n + 9n = n^2 + 10n = (n + 5)^2 - 25$$
We want $(n + 5)^2 - 25 \geq 2022 \Longrightarrow (n + 5)^2 \geq 2047$. The smallest such n for which this is the case is $n = \boxed{41}$.

8. **A**



Consider the above diagram. We can see that $LO = 2(2) = 4$ and $OZ = 2(3) = 6$. The terminal angles corresponding to Luffy and Zoro's bearings are $-15°$ and $105°$, respectively, so $m\angle LOZ = 120°$. By the law of cosines,

$$\begin{aligned} ZL^2 &= LO^2 + OZ^2 - 2(LO)(OZ)\cos(m\angle LOZ) \\ &= 4^2 + 6^2 - 2(4)(6)\left(-\frac{1}{2}\right) = \boxed{76}. \end{aligned}$$

9. **C** First, we figure out how long the rock is in the air. We solve $-16t^2 + 24t + 5 = -67 \Longrightarrow -(4t-3)^2 = -81 \Longrightarrow (4t-3)^2 = 81 \Longrightarrow t = 3, -\frac{3}{2}$. The latter makes no sense, so $t = 3$. The x-coordinate of the rock's position at time $t = 3$ is $60(3) = 180$. Since this is the edge of the opposing cliff, the distance between the cliffs is $\boxed{180}$ feet.

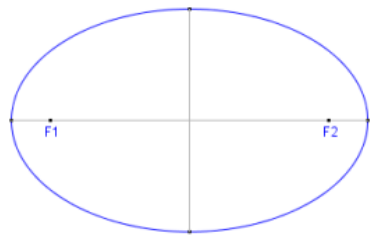
10. **C** Let Voss's height be modeled by $h(t) = A\cos\left(\frac{2\pi t}{T} + P\right) + Y$ for appropriate A, T, P, Y. First, note that $27 = Y + A$ and $15 = Y - A$, so $Y = 21$ and $A = 6$. Note that sinusoidal functions hit their maxima once per period and their minima halfway between those periods. This means that one maximum and one minimum are separated by an odd multiple of $\frac{T}{2}$. The time difference between the maximum and minimum is $123 - 1 = 122 = 2 \cdot 61$. We want this to equal $\frac{nT}{2}$ for some odd n, which gives us the possibilities $n = 1$ and $n = 61$, which correspond to periods $T = 244$ and $T = 4$, respectively. Our final answer is $6 + 244 + 4 = \boxed{254}$.

11. **C** First, we figure out how long Shirou falls. Solving $500 - 5t^2 = 0$ gives $t = \pm 10$. The negative value makes no sense here, so we take $t = 10$. Shirou fell 500 meters in 10 seconds, so his average speed is $\frac{500}{10} = 50$ m/s. Since his speed increases linearly with respect to time, his final speed is twice his average speed, or $2(50) = \boxed{100}$ m/s.

12. **B** Since α and β are acute, we can easily compute $\sin(\alpha) = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$ and $\cos(\beta) = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}$. Since they turn in opposite directions, the acute angle between their paths is $\alpha+\beta$. Using the angle addition formula gives

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\\ &= \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = \boxed{\frac{33}{65}}.\end{aligned}$$

13. **D**



Let c and a denote the distance from the center of the ellipse to either focus and the distance from the center of the ellipse to either vertex, respectively. We are looking for $\frac{c}{a}$. Suppose the Earth is at focus F_1 in the diagram above. The distance to the perigee is given by $a-c$, while the distance to the apogee is given by $a+c$. This means that

$$\frac{a-c}{a+c} = \frac{1}{100} \Longrightarrow 99a = 101c \Longrightarrow \frac{c}{a} = \boxed{\frac{99}{101}}.$$

14. **A** First, we shift our coordinate system so Rose is at the origin. This gives points $R'(0,0,0), P'(7,2,5), M'(4,2,0)$. We want $\overrightarrow{R'N'}$ to be perpendicular to $\overrightarrow{R'P'}$ and $\overrightarrow{R'M'}$, so we take the cross product to get

$$\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 2 & 5 \\ 4 & 2 & 0 \end{pmatrix} = -10\hat{\imath} + 20\hat{\jmath} + 6\hat{k}$$

Therefore, N' must be some multiple of $(-10,20,6)$. Shifting back to our original coordinate system, this means that $N = (2-10t, 20t, 1+6t)$ for some t. Of our answer choices, $\boxed{(-8,20,7)}$ corresponds to $t = 1$.

15. **D** We wish to simplify $5\sin(\theta)+6\cos(\theta)$. Taking inspiration from the sine addition formula, we multiply and divide by $\sqrt{5^2+6^2}=\sqrt{61}$ to get

$$\begin{aligned} 5\sin(\theta)+6\cos(\theta) &= \sqrt{61}\left(\frac{5}{\sqrt{61}}\sin(\theta)+\frac{6}{\sqrt{61}}\cos(\theta)\right) \\ &= \sqrt{61}\sin\left(\theta+\arccos\left(\frac{5}{\sqrt{61}}\right)\right) \end{aligned}$$

This reduces our graph to that of $r=8+\sqrt{61}\sin\left(\theta+\arccos\left(\frac{5}{\sqrt{61}}\right)\right)$, which is a rotated version of $r=8+\sqrt{61}\sin(\theta)$. Since $\sqrt{61}<8<2\sqrt{61}$, this graph is a $\boxed{\text{dimpled}}$ limaçon.

16. **E** Let n denote the number of susuwatari in the boiler room. In the language of modular arithmetic, the given information becomes

$$\begin{aligned} n &\equiv 5 \pmod 6 \\ n &\equiv 6 \pmod 7 \\ n &\equiv 7 \pmod 8. \end{aligned}$$

These can be written as $n\equiv -1 \pmod 6$, $n\equiv -1 \pmod 7$, and $n\equiv -1 \pmod 8$, respectively. This means that $n\equiv -1\ (\mathrm{mod}\ \mathrm{lcm}(6,7,8)=168)$. n must be non-negative, so our smallest possible n is $n=167$, which has sum of digits $1+6+7=\boxed{14}$

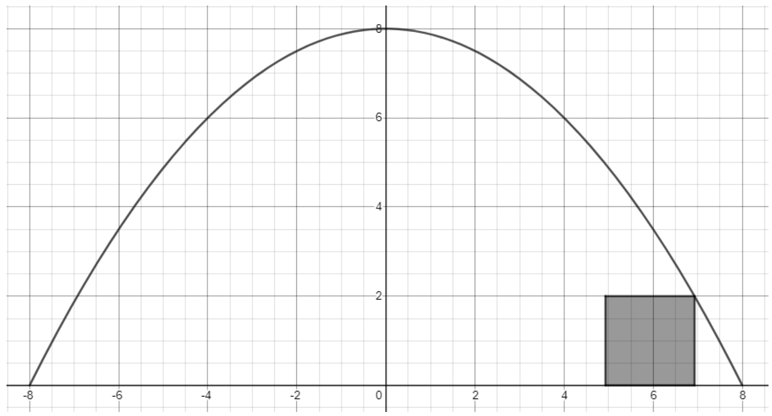
17. **D** Let M_n and B_n denote the number of mature and baby carrots on day n. We can construct the recurrences

$$\begin{aligned} M_n &= M_{n-1}+B_{n-1} \\ B_n &= M_{n-1}. \end{aligned}$$

Substituting the second into the first gives $M_n=M_{n-1}+M_{n-2}$. We know that $M_0=1$ and $M_1=1$, so we can easily see that $M_n=F_{n+1}$ for all n. Plugging in $n=30$ gives our desired answer of $\boxed{F_{31}}$.

18. **C** From any given vertex of the hexagon, there are 2 vertices a distance 1 away, 2 vertices a distance $\sqrt{3}$ away, and 1 vertex a distance 2 away. Each of these vertices are equally likely. As a result, the expected distance Bill will run on any given shine of the pointer is $\frac{2}{5}(1)+\frac{2}{5}\left(\sqrt{3}\right)+\frac{1}{5}(2)=\frac{4+2\sqrt{3}}{5}$. By linearity of expectation, the expected total distance Bill runs is $50\left(\frac{4+2\sqrt{3}}{5}\right)=\boxed{40+20\sqrt{3}}$.

19. **A**



Since the entire car moves as one unit, the actual location of the center of mass does not matter. WLOG, let it be at the geometric center of the square. The furthest right the car can go is the point where its upper right corner touches the side of the tunnel. This contact point is at $y = 2$. Plugging $y = 2$ into $x^2 + 8y = 64$ gives $x^2 = 48 \Longrightarrow x = \pm 4\sqrt{3}$. We are looking for the rightmost point, so we take the positive value. Now, the center of mass of the car is one unit to the left of this at $x = 4\sqrt{3} - 1$. Applying the same reasoning to the left side, the furthest left the center of mass of the car can be is $x = -(4\sqrt{3} - 1)$, yielding a total interval of length $\boxed{8\sqrt{3} - 2}$.

20. **B** The initial turn by angle θ takes Michiru from an angle of $\arctan\left(\frac{1}{3}\right)$ to an angle of $\arctan\left(\frac{2}{5}\right)$, so $\theta = \arctan\left(\frac{2}{5}\right) - \arctan\left(\frac{1}{3}\right) = \arctan\left(\frac{\frac{2}{5}-\frac{1}{3}}{1+\left(\frac{2}{5}\right)\left(\frac{1}{3}\right)}\right) = \arctan\left(\frac{1}{17}\right)$. The next rotation by θ takes Michiru from an angle of $\arctan\left(\frac{2}{5}\right) + \arctan\left(\frac{1}{17}\right) = \arctan\left(\frac{\frac{2}{5}+\frac{1}{17}}{1-\left(\frac{2}{5}\right)\left(\frac{1}{17}\right)}\right) = \arctan\left(\frac{39}{83}\right)$. This means the next point Michiru passes will be 39 blocks in the $+y$ direction and 83 blocks in the $+x$ direction from her current location at $(8,3)$, which is $(\boxed{91}, 42)$.

21. **C** Using the angle sum identities, we find that Shigeo's and Teruki's power levels are

$$25\sin\left(t + \arcsin\left(\frac{7}{25}\right)\right) = 25\left(\frac{24}{25}\sin(t) + \frac{7}{25}\cos(t)\right) = 24\sin(t) + 7\cos(t)$$

and

$$15\cos\left(t + \arccos\left(\frac{3}{5}\right)\right) = 15\left(\frac{3}{5}\cos(t) - \frac{4}{5}\sin(t)\right) = 9\cos(t) - 12\sin(t),$$

respectively. Adding these together gives a combined power of $16\cos(t) + 12\sin(t)$, which is equivalent to

$$20\left(\frac{4}{5}\cos(t) + \frac{3}{5}\sin(t)\right) = 20\sin\left(t + \arccos\left(\frac{3}{5}\right)\right),$$

Which clearly has maximum value $\boxed{20}$

22. **E** Note that for any positive integer n, the matrix

$$\begin{pmatrix} n+6 & 3 \\ 2 & 1 \end{pmatrix}$$

Has distinct positive integer entries and determinant n. Therefore, there is $\boxed{\text{no such n}}$ for which no matrix exists.

23. **C** The first two terms of the expression are equivalent to

$$\begin{aligned}\sin(2\theta)\cos^2(2\theta) - \sin^3(2\theta) &= \sin(2\theta)(\cos^2(2\theta) - \sin^2(2\theta)) \\ &= \sin(2\theta)\cos(4\theta).\end{aligned}$$

The latter two terms of the expression are equivalent to

$$\begin{aligned}\cos^2(\theta)\sin(4\theta) - \sin^2(\theta)\sin(4\theta) &= \sin(4\theta)(\cos^2(\theta) - \sin^2(\theta)) \\ &= \sin(4\theta)\cos(2\theta).\end{aligned}$$

Adding these together, we get

$$\sin(2\theta)\cos(4\theta) + \sin(4\theta)\cos(2\theta) = \sin(6\theta).$$

The graph of $r = \sin(6\theta)$ is a rose curve with $\boxed{12}$ petals.

24. **A** Let A denote the event that the card actually has a 1 on its face, and R denote the event that Rei says the card has a 1 on its face. We are looking for $P(A|R)$. By Baye's rule, this is equivalent to

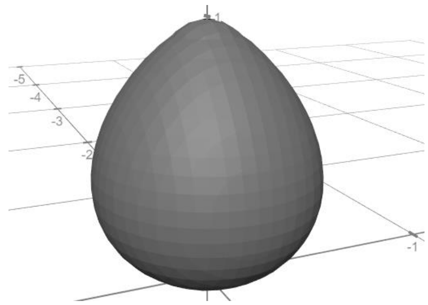
$$P(A|R) = \frac{P(R|A)P(A)}{P(R|A)P(A) + P(R|A')P(A')}.$$

We can easily calculate that $P(A) = \frac{1}{6}$, $P(A') = \frac{5}{6}$, $P(R|A) = \frac{62}{100}$, and $P(R|A') = \frac{1}{2}\left(\frac{38}{100}\right) = \frac{19}{100}$. Plugging these into our Baye's rule expression gives

$$P(A|R) = \frac{\frac{62}{100}\left(\frac{1}{6}\right)}{\frac{62}{100}\left(\frac{1}{6}\right) + \frac{19}{100}\left(\frac{5}{6}\right)} = \boxed{\frac{62}{157}}.$$

25. **C** Jack wants to maximize the probability that $2^n\theta$ is in the second or fourth quadrants. Note that if $n = 0$, then $\tan(\theta)$ is necessarily positive, so Jack always loses. Note that if $n > 0$, Jack can pick $\theta = \frac{\pi}{2} - \frac{1}{2^{2022}}$ (or anything else sufficiently close to but less than $\frac{\pi}{2}$), yielding $2\theta = \pi - \frac{1}{2^{2021}}$, $4\theta = 2\pi - \frac{1}{2^{2020}}$, $8\theta = 4\pi - \frac{1}{2^{2019}}$, and $16\theta = 8\pi - \frac{1}{2^{2018}}$ all being in the second or fourth quadrants. This is clearly optimal, as Jack wins every case where winning is theoretically possible. The probability of getting $n = 0$ is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, so Jack wins with probability $\boxed{\frac{15}{16}}$.

26. **B**



Converting to cylindrical coordinates gives $r^2 = z(1-z)^{3/2}$, so the egg is radially symmetric about the z-axis and all cross sections are circles. Our objective is thus to maximize πr^2, which is equivalent to maximizing $z(1-z)^{3/2}$. Now, by the AM-GM inequality,

$$\frac{1}{5} = \frac{\frac{z}{2}+\frac{z}{2}+\frac{1-z}{3}+\frac{1-z}{3}+\frac{1-z}{3}}{5} \geq \sqrt[5]{\frac{z^2(1-z)^3}{2^2\cdot 3^3}}.$$

This implies that

$$z^{2/5}(1-z)^{3/5} \leq \frac{2^{2/5}\cdot 3^{3/5}}{5}.$$

Raising both sides to the power of $5/2$ gives

$$z(1-z)^{3/2} \leq \frac{2\cdot 3^{\frac{3}{2}}}{5^{\frac{5}{2}}} = \frac{6\sqrt{15}}{125}.$$

This can be attained by setting $z = \frac{2}{5}$. Therefore, our maximum area is $\frac{6\pi\sqrt{15}}{125}$, yielding a sum of $6 + 15 + 125 = \boxed{146}$.

27. **D** The given information can be encoded as

$$T\begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 6 \\ 4 & -1 & -3 \\ 7 & 4 & -1 \end{pmatrix}$$

To solve for T, we need to right-multiply both sides by $\begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$. To find this inverse, we first take the cofactor matrix, getting

$$\begin{pmatrix} -2 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & -1 \end{pmatrix}.$$

Next, we need to take the transpose of this and divide by

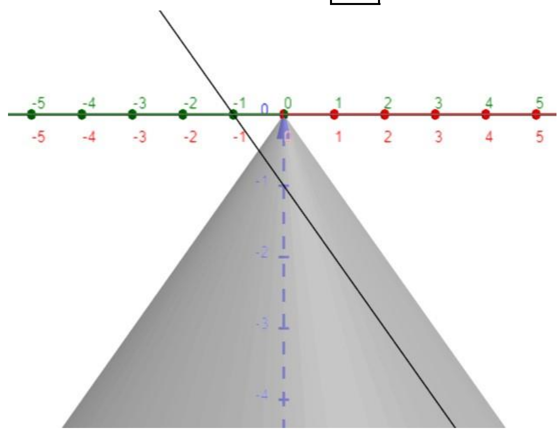
$$\det\begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -1$$

to get

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Finally, we can multiply to get
$$T=\begin{pmatrix}-1 & 1 & 6\\ 4 & -1 & -3\\ 7 & 4 & -1\end{pmatrix}\begin{pmatrix}2 & 1 & 1\\ -1 & -1 & 0\\ 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}3 & 4 & 5\\ 6 & 2 & 1\\ 9 & 2 & 6\end{pmatrix}$$
The location of the portal after the transformation is then
$$\begin{pmatrix}3 & 4 & 5\\ 6 & 2 & 1\\ 9 & 2 & 6\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}=\begin{pmatrix}12\\ 9\\ 17\end{pmatrix},$$
which has sum of coordinates $12+9+17=\boxed{38}$

28. **D**



The conic section formed depends on the relative "steepness" of the plane and the cone. If the plane is completely flat, we get a circle; if the plane is less steep than the cone, we get an ellipse; if the plane is equally steep, we get a parabola; and if the plane is steeper, we get a hyperbola (or half a hyperbola in this case, since we're only considering a single infinite cone rather than a double-napped cone). To quantify "steepness," consider the angle between the surface and the xy-plane.

To find the steepness of the cone, fix $y=0$ to get $2x^2=z^2 \Longrightarrow z=\pm\sqrt{2}x$. Thus, the angle that the cone makes with the xy-plane is $\arctan\left(\sqrt{2}\right)$.

To find the steepness of the plane, first take the normal vector $\vec{N}=\langle 1,1,1\rangle$ and the corresponding vector $\vec{p}=\langle 1,1,0\rangle$ in the xy-plane. We can compute that the cosine of the angle between $\vec{N}$ and $\vec{p}$ is
$$\cos(\theta)=\frac{|\vec{N}\cdot\vec{p}|}{||\vec{N}||\cdot||\vec{p}||}=\frac{2}{\sqrt{2}\cdot\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{3}},$$
so $\theta=\arccos\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$. To find the angle between the plane and the xy-plane, we simply take $\frac{\pi}{2}-\arccos\left(\frac{\sqrt{2}}{\sqrt{3}}\right)=\arcsin\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$. Drawing a right triangle, we can see that this is equal to $\arctan(\sqrt{2})$.

Since the plane and cone have equal steepness, the conic section is a $\boxed{\text{parabola}}$.

29. **A** Let M_n denote the number of sequences of houses with no two consecutive houses receiving letters and n being the largest number among any of the houses that receive letters. We can derive the recurrence

$$M_n = M_{n-2} + M_{n-3} + \cdots + M_1 + M_0,$$

Where M_0 denotes the case where no houses receive letters. Plugging in $n+1$ to this, we see that

$$M_{n+1} = M_{n-1} + M_{n-2} + \cdots + M_1 + M_0 = M_{n-1} + M_n$$

We can easily compute that $M_1 = M_2 = 1$, so $M_n = F_n$. Now, we wish to compute

$$S = F_3 + F_6 + F_9 + \cdots + F_{30}.$$

First, use $F_n = F_{n-1} + F_{n-2}$ to get

$$S = F_1 + F_2 + F_4 + F_5 + F_7 + F_8 + \cdots + F_{28} + F_{29}.$$

Adding these together, we get

$$2S = F_1 + F_2 + F_3 + F_4 + \cdots + F_{30}.$$

Finally, use $F_n = F_{n+2} - F_{n+1}$ to get

$$2S = (F_3 - F_2) + (F_4 - F_3) + (F_5 - F_4) + \cdots + (F_{32} - F_{31}),$$

Which simplifies to $S = \frac{F_{32} - F_2}{2} = \boxed{\frac{F_{32} - 1}{2}}$.

30. **B** Note that $V^2 + 3V + 2 = (V+1)(V+2)$, so the probability of flip k landing heads is $(2^{k-1}+1)(2^{k-1}+2)$. Our expected value thus becomes

$$E_n = \sum_{k=1}^{n} \frac{2^{k-1}}{(2^{k-1}+1)(2^{k-1}+2)} = \sum_{k=0}^{n-1} \frac{2^k}{(2^k+1)(2^k+2)}$$

Dividing top and bottom of the summand by 2 gives

$$\frac{2^{k-1}}{(2^k+1)(2^{k-1}+1)} = \frac{(2^k+1)-(2^{k-1}+1)}{(2^k+1)(2^{k-1}+1)} = \frac{1}{2^{k-1}+1} - \frac{1}{2^k+1},$$

so our sum becomes

$$E_n = \sum_{k=0}^{n-1} \left(\frac{1}{2^{k-1}+1} - \frac{1}{2^k+1} \right) = \frac{1}{2^{-1}+1} - \frac{1}{2^{n-1}+1}.$$

As n grows large, this second term vanishes, so E_n approaches $\frac{1}{2^{-1}+1} = \boxed{\frac{2}{3}}$.