# Alpha

# Equations and Inequalities

# Test #422

Directions:

1. Fill out the top left section of the scantron. Do not abbreviate your school name.

2. In the Student ID Number grid, write your 9-digit ID# and bubble.

3. In the Test Code grid, write the 3-digit test# on this test cover and bubble.

4. Scoring for this test is 5 times the number correct plus the number omitted.

5. TURN OFF ALL CELL PHONES.

6. No calculators may be used on this test.

7. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

8. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

9. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

10. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

11. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

12. Unless a question asks for an approximation or a rounded answer, give the exact answer.

1. How many distinct integer solutions are there to the equation $x^{2022} = 1$?
   A. 0 B. 1 C. 2 D. 2022 E. NOTA

2. How many positive integer solutions are there to $x^2 + 8x - 65 \leq 0$?
   A. 4 B. 5 C. 6 D. 7 E. NOTA

3. What is the sum of the distinct roots of $x^4 - 2x^3 - 19x^2 + 68x = 60$?
   A. $-2$ B. 0 C. 2 D. 5 E. NOTA

4. For $0 \leq \theta < 2\pi$, what is the sum of solutions for which $\sin(2\theta) = \cos(\theta)$.
   A. $\pi$ B. $\frac{3\pi}{2}$ C. $2\pi$ D. $3\pi$ E. NOTA

5. $3^{2x+y} = 729$. $2^{3x-y} = 16384$. What is $x + y$?
   A. $\frac{9}{5}$ B. 2 C. $\frac{16}{5}$ D. 4 E. NOTA

6. $\sqrt{40 + 42i}$ can be expressed in the form $a + bi$ where $a$ and $b$ are integers. What is $a + b$ where $a > 0$?
   A. 3 B. 4 C. 7 D. 10 E. NOTA

7. $\log_2(x + 3) + \log_4(x^2 + 8x + 16) = 1$. What is the sum of solutions of x?
   A. $-7$ B. $-2$ C. 3 D. 5 E. NOTA

8. How many petals are on the rose curve $r = 6\sin(24\theta)$
   A. 24 B. 36 C. 48 D. 96 E. NOTA

9. Let $x_1, x_2, \ldots x_6$ be some permutation of the numbers 1,2,4,5,7,10. What is the minimum value of $\sum_{n=1}^{6}(x_n + 2n)^2$

A. 829 B. 843 C. 883 D. 1087 E. NOTA

10. What is the sum of the reciprocal of the roots of $10x^4 - 7x^3 - 5x^2 + x + 1$?

A. $-1$ B. $\frac{7}{10}$ C. 1 D. $\frac{10}{7}$ E. NOTA

11. $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} M = \begin{bmatrix} 23 & 7 \\ 31 & -3 \end{bmatrix}$ where M is a 2x2 matrix. What is the sum of the elements of M?

A. 1 B. 7 C. 12 D. 14 E. NOTA

12. There are 3 numbers $a, b, c$. The harmonic means of the numbers is 2, the geometric mean is 6, and the arithmetic mean is 10. What is the value of $a^2 + b^2 + c^2$?

A. 252 B. 364 C. 648 D. 684 E. NOTA

13. $\cos^2\theta - 1 = \frac{-\sin\theta - 4}{6}$. What is the sum of the solutions for $0 \leq \theta < 2\pi$

A. 0 B. $2\pi$ C. $3\pi$ D. $4\pi$ E. NOTA

14. $x^2 + xy + xz = 50$
$y^2 + yx + yz = 40$
$z^2 + zx + zy = 10$
What is $x + y + z$, where $x, y, z$ are real numbers and $x > 0$?

A. 0 B. 5 C. 10 D. 15 E. NOTA

15. How many ordered triplets $(x, y, z)$ of real numbers satisfy the following equations
$z - x - y = -3$
$xy - yz - xz = -24$
$xyz = 80$
A. 0 B. 3 C. 6 D. 27 E. NOTA

16. What conic is formed by the equation $\sqrt{(x-5)^2 + (y-3)^2} + \sqrt{(x+5)^2 + (y+7)^2} = 15$
A. Circle B. Ellipse C. Hyperbola D. Parabola E. NOTA

17. $x^2 + 3x + 4 = 0$. What is $(x+1)(x+4)^2(x+7)$?
A. $-64$ B. $-48$ C. $-32$ D. $-16$ E. NOTA

18. Let $a_1, a_2, \ldots a_n$ be real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 1$.
What is the minimum value $n$ such that $a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2$ can equal $\frac{1}{50}$.
A. 25 B. 50 C. 100 D. 2500 E. NOTA

19. Let $p, q$ be the smallest positive integers such that $(1+i)^p = \left(\sqrt{3} + i\right)^q$. What is $p + q$?
A. 12 B. 18 C. 24 D. 36 E. NOTA

20. Suppose an economy only has two sectors, Goods and Services. The Goods sector sells 70% of its outputs to Services and keeps the rest. The Services sector sells 67% of its outputs to the Goods sector and keeps the rest. Find the equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures. Let $p_g$ and $p_s$ denote the total dollar values of the total annual outputs of the Goods and Services sectors, respectively. Given $p_s = 700$, what is $p_g$ rounded to the nearest integer?
A. 300 B. 330 C. 670 D. 700 E. NOTA

21. Let $a_1, a_2, a_3, a_4$ be the roots of the polynomial $x^4 - 6x^3 - 4x^2 + 15x - 20$.
What is $a_1^3(a_2 + a_3 + a_4) + a_2^3(a_1 + a_3 + a_4) + a_3^3(a_1 + a_2 + a_4) + a_4^3(a_1 + a_2 + a_3)$?

A. -106 B. -86 C. -6 D. 86 E. NOTA

22. How many ordered triples $(x, y, z)$ of real numbers satisfy the $x^4 + y^4 + z^4 + 16 = 8xyz$?

A. 4 B. 8 C. 12 D. 16 E. NOTA

23.
$$\sqrt{x+2} + \sqrt{x+3} + \sqrt{x+4} = \sqrt{y-3} + \sqrt{y-4} + \sqrt{y-5}$$
$$y^2 - 2y - x^2 + x = 126$$

What is $x + y$?

A. $\frac{77}{13}$ B. 14 C. 18 D. 21 E. NOTA

24. The line $\frac{x-1}{2} = \frac{y-3}{-3} = z + 4$ and the plane $13x + 3y + 9z = 47$ intersect at the point (a,b,c).
What is a+b+c?

A. $-\frac{37}{13}$ B. 0 C. $\frac{5}{13}$ D. $\frac{47}{26}$ E. NOTA

25. How many of the following inequalities are always true?

I. $|a + b| \leq |a| + |b|$ for 2 real numbers $a, b$.
II. $|u + v| \leq |u| + |v|$ for two vectors $u$ and $v$.
III. $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \geq {a_1}^{\lambda_1} a_2^{\lambda_2} \ldots {a_n}^{\lambda_n}$ where $a_1, a_2, \ldots, a_n$ are nonnegative real numbers and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.
IV. $a^4 + b^4 + c^4 \geq a^3 b + b^3 c + c^3 a$ for any nonnegative real number $a, b, c$.

A. 0 B. 1 C. 2 D. 3 E. NOTA

26. Let $\vec{u} =< 2, -1, 2 >$. Let $\vec{v} =< 3, 5, 2 >$. Decompose vector $v$ into two vectors, $\overrightarrow{v_1}$, which is orthogonal to $\vec{u}, and\ \overrightarrow{v_2}$ which is parallel to $\vec{u}$. What is the sum of the components of $\overrightarrow{v_1}$? $(\overrightarrow{v_1} + \overrightarrow{v_2} = \vec{v})$

A. $\frac{5}{3}$ B. $\frac{25}{9}$ C. $\frac{360}{41}$ D. 9 E. NOTA

27. What is the maximum value of $f(x, y) = x^2y^2$ subject to the ellipse $6x^2 + y^2 = 12$

A. $\sqrt{6}$ B. $\sqrt{12}$ C. 6 D. 12 E. NOTA

28. How many solution sets (x,y,z) do the following system of 3 equations have?

$$3x + 3y + 4z = 0$$
$$5x + 7y + 3z = 0$$
$$2x + 4y - z = 0$$

A. 0 B. 1 C. 2 D. $\infty$ E. NOTA

29. On an island, there are three types of people, knights, knaves, and spies. Knights can only tell the truth, knaves can only lie, and spies can either tell the truth or lie. On the island, you meet Eddie, Jack, and Andrew. One of them is a spy, one of them is a knight, and one of them is a knave. They all know each other's roles. Andrew says "I am a spy", Eddie says "I am a knight", and Jack says "I am a knave". Who is the knave?

A. Andrew B. Eddie C. Jack D. Not enough information E. NOTA

30. $x = 2022!$. What is the unit digit of x?

A. 0 B. 1 C. 2 D. 2022 E. NOTA