Mu

Applications

Test #431

Directions:

1. Fill out the top left section of the scantron. Do not abbreviate your school name.

2. In the Student ID Number grid, write your 9-digit ID# and bubble.

3. In the Test Code grid, write the 3-digit test# on this test cover and bubble.

4. Scoring for this test is 5 times the number correct plus the number omitted.

5. TURN OFF ALL CELL PHONES.

6. No calculators may be used on this test.

7. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

8. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

9. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

10. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

11. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

12. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Unless otherwise specified, the domain and range of functions are limited to the real numbers. "NOTA" stands for "None Of These Answers." Read all questions carefully. Good luck!

1. Yared drives his Range Rover Sports Truck on the Cartesian plane according to the equations $x = \frac{3t^2}{2}$ and $y = \frac{t^3}{3}$. Find the total distance he travels from $t = 0$ to $t = 6\sqrt{2}$.

   A. $108 + 144\sqrt{2}$ B. $36\sqrt{113}$ C. 243 D. 234 E. NOTA

2. Farmer Dhilan wants to build a triangular pen using 512 ft of fencing. He will use a straight cliffside as one side of the triangle. Find the maximum area his pen can enclose, in ft^2.

   A. $2^{14}\sqrt{3}$ B. ${}^{2^{18}}/_{9}$ C. 2^{15} D. 2^{16} E. NOTA

3. Amy's acceleration at time t is given by the vector $a(t) = \langle 4t, 9t^2 - 3\rangle$. If she is at rest at $t = 0$, calculate Amy's speed at $t = 1$.

   A. Not enough information B. 4 C. $2\sqrt{85}$ D. 2 E. NOTA

4. The ellipse $(x + 6)^2 + 4(y - 3)^2 = 16$ forms the base of a hat. Cross-sections of the hat taken perpendicular to the x-axis are equilateral triangles. (Assume that the elliptical base is a hole and that the sides of the hat are infinitely thin.) Matt wants to turn the hat over and fill it up exactly to the top with hummus. What is the total volume of hummus that Matt needs?

   A. $\frac{16\sqrt{3}}{3}$ B. $\frac{32\sqrt{3}}{3}$ C. $\frac{44\sqrt{3}}{3}$ D. $\frac{64\sqrt{3}}{3}$ E. NOTA

5. The cylinder with the greatest volume that can be inscribed in a sphere of radius length R has a height of $11\sqrt{3}$. Find R.

   A. 33 B. $33/2$ C. $33\sqrt{2}/2$ D. $11\sqrt{6}/2$ E. NOTA

6. A sample of Urpayneum undergoes exponential decay, with a half-life of 20 minutes. After how many minutes will exactly 1% of the original amount of Urpayneum remain?

   A. $40\log_2 10$ B. $90\log_{10} 2$ C. $24/5$ D. $198/5$ E. NOTA

7. In an ecosystem, the population y of fish people and the population x of magic kelp (their primary food source), both in millions, are related by the differential equation

$$\frac{dy}{dx} = \frac{5}{3xy+y+6x+2}.$$

If initially $x = 1$ and $y = 2$, which of the following is a possible value for the population of magic kelp when $y = 4$, in millions? (Assume fractional or irrational values are allowed.)

A. $\dfrac{e^{48/5}-1}{3}$ B. $\dfrac{4e^2-1}{3}$ C. $\dfrac{4e^6-1}{3}$ D. $\dfrac{2\sqrt{7}-1}{3}$ E. NOTA

8. Find the total distance traveled from time $t = 2$ to $t = 6\sqrt{3}$ by a train whose speed v is given by

$$v(t) = \frac{1}{\left(\frac{t}{16}\right)^{\frac{2}{3}}\sqrt{1-\left(\frac{t}{16}\right)^{\frac{2}{3}}}}.$$

A. $\pi/6$ B. $8\pi/3$ C. 8π D. 16π E. NOTA

9. Remy combines two sauces with respective butter concentrations 20% and 50% to make 100 mL of a sauce with a 41% butter concentration. Find the positive difference of the volumes of the two sauces used, in mL.

A. 40 B. 50 C. 60 D. 70 E. NOTA

10. Ann's math team voted her the Most Valuable Trigonometer (MVT). She travels from $x = 2$ to $x = 5$ along a continuous and differentiable function $f(x)$. If $f(2) = 14\pi$ and $f(5) = 20\pi$, which of the following must her slope $f'(x)$ equal at some $x = c$, where $2 < c < 5$?

A. 0 B. $\dfrac{7}{2}$ C. 2π D. $\dfrac{1}{2\pi}$ E. NOTA

11. At time $t = 0$, Sharay enters a car on the circumference of a Ferris wheel at the lowest point on the wheel. As the wheel rotates counterclockwise, Sharay's height h is given by $h(t) = 70 + 50\sin(\frac{\pi}{9}t + \frac{3\pi}{2})$. In m/min, at what instantaneous rate is Sharay's height changing the **second time** her height is 45 m?

A. $\dfrac{25\pi}{9}$ B. $-\dfrac{25\pi}{9}$ C. $\dfrac{25\pi\sqrt{3}}{9}$ D. $-\dfrac{25\pi\sqrt{3}}{9}$ E. NOTA

12. A circular ripple in a pond, centered at $(1, 5)$, expands outward such that its radius increases at a constant rate of 24 units/s. When the ripple's greater x-intercept is 13, at what rate is the value of this x-intercept increasing?

A. $\frac{13}{12}$ B. 24 C. $\frac{599}{13}$ D. $\frac{24\sqrt{194}}{13}$ E. NOTA

13. A sphere centered at the origin has a **volume** that increases at a constant rate of $8\ units^3/s$. (Assume that as the sphere grows, it remains a sphere centered at the origin.) Let A be the area of the cross section made by the sphere's intersection with the plane $2x + y - 2z = 9$. What is the rate of change of A the instant the sphere's radius is 5?

A. $4/5$ B. $16/5$ C. $16/25$ D. 8π E. NOTA

14. Let N be the sum of the nonreal roots of $x^3 - 3x^2 + 3x - 2023 = 0$. Find $\lfloor N \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

A. -11 B. -20 C. 2 D. 3 E. NOTA

15. Mr. Moody's anger A is given by

$$A(n) = \frac{n + p + \frac{1}{n+p} - (n + \frac{1}{n})}{p}$$

where n is the number of students in his class, for a given patience level p. Let $L(n)$ be the limit of Mr. Moody's anger as p approaches 0. Find $L(21)$.

A. $\frac{440}{441}$ B. 1 C. $\frac{442}{441}$ D. $\frac{442}{21}$ E. NOTA

16. A computer-aided design program forms a 3D shape by revolving the graph of $y = x\sqrt{|\sin x|}$ around the x-axis, from $x = 0$ to $x = 2\pi$. What is the total volume of this shape?

A. $4\pi^2$ B. $4\pi^3$ C. $6\pi^3 - 8\pi$ D. $20\pi^4 - 48\pi^2$ E. NOTA

17. The minute and hour hands of a clock point in the directions $\langle 3, 2, 6\rangle$ and $\langle 4, 0, 3\rangle$, respectively. Find the cosine of the smaller angle between them.

A. $\frac{13}{35}$ B. $\frac{\sqrt{13}}{7}$ C. $\frac{6}{7}$ D. 30 E. NOTA

18. In the study of differential equations, the Laplace transform $\mathcal{L}[f](s)$ of a function $f(t)$ is given by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt$$

For $g(t) = 2\cos t\,(\sec t + \sin t)$, let $G(s) = \mathcal{L}[g](s)$. Find $G(2)$.

A. -2 B. $3/4$ C. $5/4$ D. $76/17$ E. NOTA

19. Let $M(a, n)$ be a function that returns the nth smallest positive multiple of a positive integer a. Gage uses a computer program with two loops to (correctly) compute

$$S = \sum_{k=1}^{13} \sum_{j=1}^{200} M(k, j)$$

What is the sum of the digits of S?

A. 28 B. 26 C. 21 D. 12 E. NOTA

For questions 20 *and* 21, *a rope is* 100 *m long and has uniform density* 1 *kg/m. Use* $10\ m/s^2$ *as the acceleration due to gravity.*

20. Kasra and Milaan live on different floors of a very tall apartment building. Kasra drops one end of the rope from his window. Milaan, 100 m below, sticks his arm out his window and grabs the end of the rope, so that the rope is now taut between them. Kasra then lets go of his end, and the rope swings down to hang from Milaan's hand. What is the work done by gravity on the rope, in $kg \cdot m^2/s^2$, after Kasra completely lets go of the rope?

A. $300{,}000$ B. $100{,}000$ C. $50{,}000$ D. $10{,}000$ E. NOTA

21. A passerby now attaches a leaking bucket to the bottom end of the hanging rope, and Milaan hoists it up a distance of 100 m to his window at a constant rate. When empty, the bucket has a mass of 5 kg. When the bucket begins its ascent, it holds 35 kg of caramel, which leaks out at a rate of 2 kg/s. If it takes 12.5 seconds to lift the bucket and rope the full distance, what is the total work done by Milaan, in $kg \cdot m^2/s^2$?

A. $\frac{4{,}900{,}000}{3}$ B. $127{,}500$ C. $90{,}000$ D. $77{,}500$ E. NOTA

22. A perfectly spherical hailstone melts (decreases in volume) at a rate proportional to its surface area. Initially, it has a radius of 8 cm, and after 4 minutes its radius is 6 cm. What is its radius in cm after another 4 minutes? Assume the hail stone remains spherical at all times.

A. $2\sqrt{2}$ B. 4 C. $3\sqrt{2}$ D. $9/2$ E. NOTA

23. Rick makes an ice cream cone by revolving the segment of the line $x + 2y = 2$ in the first quadrant around the x-axis. Find the volume his cone can hold, assuming a negligible thickness.

A. $\frac{\pi}{3}$ B. $\frac{7\pi}{12}$ C. $\frac{4\pi}{3}$ D. 2π E. NOTA

24. Kyle begins at $(0, -3)$ and runs along the line $y = -3$ so that his x-coordinate is increasing at a constant rate of 1 unit/s. A second person, incidentally also named Kyle, runs along the graph of $y = f(x) = -\frac{1}{40}x^4 + \frac{3}{20}x^3 + \frac{3}{5}x^2 + \frac{7}{5}x + \frac{17}{4}$ such that both Kyles always share the same x-coordinate. The distance between the Kyles measure d units. At the inflection point of $f(x)$ where $x > 0$, Kyle will pass a football to Kyle. At what rate is d changing, in units/s, when the pass is made? Assume the pass takes 0 seconds to complete.

A. 4 B. $5\sqrt{2}$ C. 7 D. 10 E. NOTA

25. The rate (in mL/s) at which apple tea is poured into a glass while the time $t \geq 1$ is given by

$$r(t) = \frac{3t^7+14t^6+3t+2}{t^7+t}.$$

If at $t = 1$ the glass contained 22 mL of tea, then the volume of tea in at $t = 2$ is $a + b\ln c$ mL, for positive integers a, b, and c, with c square-free. What is $a + b + c$?

A. 89 B. 90 C. 91 D. 92 E. NOTA

26. Al gives his favorite cubic polynomial $a(x)$ to Betty. She observes that the first two terms are $x^3 - 6x^2$, but the last two terms $cx + d$ are smudged and Betty cannot read the coefficients. However, when Al tells Betty that $a(x)$ has three real, nonnegative roots, Betty realizes that the constant term d has a **minimum** value m. Find m.

A. -8 B. -6 C. 0 D. 8 E. NOTA

27. At the National Urns and Balls Championship, Zach and Lou take turns drawing a black or white ball from an urn at random. If a player draws a black ball, he wins. If a player draws a white ball, he must replace it and then add 2 black balls to the urn (from an infinite stockpile) for his opponent's turn. The urn initially contains exactly 1 black ball and 1 white ball, and Zach will draw first. What is the probability Zach will win?

A. $\sqrt{e}/e$ B. $\sinh\left(\frac{1}{2}\right)$ C. $1-\sinh\left(\frac{1}{2}\right)$ D. $\frac{4}{5}-\tan^{-1}\left(\frac{1}{2}\right)$ E. NOTA

28. The population P of porcupines in a forest is initially 50 and undergoes logistic growth, described by the equation $\frac{dP}{dt}=4P(1-\frac{P}{400})$. At what value of P is the population growing the fastest?

A. 100 B. 50 C. 400 D. 200 E. NOTA

29. How many distinct permutations of the letters in HEEBEEJEEBEES **do not** begin with E?

A. $29{,}700$ B. $35{,}640$ C. $59{,}400$ D. $77{,}220$ E. NOTA

30. Sammy skates along the graph of

$$2|z|^7=z^6+(\,\overline{z}\,)^6$$

as θ increases in the Argand plane, where z is a complex number $r\cdot(\cos\theta+i\sin\theta)$, $i=\sqrt{-1}$, and $\overline{z}$ is the complex conjugate of z. She begins at the point in the Argand plane representing the number 1. How many times will Sammy pass through the origin before she returns to her starting point?

A. 2 B. 6 C. 11 D. 12 E. NOTA