

Mu

Combinatorics and Probability

Test #433

Directions:

1. Fill out the top left section of the scantron. Do not abbreviate your school name.
2. In the Student ID Number grid, write your 9-digit ID# and bubble.
3. In the Test Code grid, write the 3-digit test# on this test cover and bubble.
4. Scoring for this test is 5 times the number correct plus the number omitted.
5. TURN OFF ALL CELL PHONES.
6. No calculators may be used on this test.
7. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.
8. If a student believes a test item is defective, select “E) NOTA” and file a dispute explaining why.
9. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.
10. If a problem has wording like “which of the following could be” or “what is one solution of”, an answer choice providing one of the possibilities is considered to be correct. Do not select “E) NOTA” in that instance.
11. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select “E) NOTA” in that instance.
12. Unless a question asks for an approximation or a rounded answer, give the exact answer.

For all questions, the answer choice "E. NOTA" means none of the above answers is correct. The notation $[n]$ represents the set of integers $\{1, 2, \dots, n\}$. Good luck and have fun!

1. What is the probability of obtaining exactly 4 tails in 5 flips of a fair coin?
A. $\frac{1}{16}$ B. $\frac{3}{32}$ C. $\frac{1}{8}$ D. $\frac{5}{32}$ E. NOTA
2. Two six-sided dice are rolled. What is the probability that the sum of the numbers rolled is less than 11?
A. $\frac{8}{9}$ B. $\frac{11}{12}$ C. $\frac{17}{18}$ D. $\frac{35}{36}$ E. NOTA
3. How many ways can 9 students be divided into 3 groups of 3?
A. 35 B. 140 C. 280 D. 1680 E. NOTA
4. A bag contains 4 black, 3 pink, and 3 blue marbles. If three marbles are drawn at random without replacement, what is the probability that all three are of different colors?
A. $\frac{1}{20}$ B. $\frac{1}{12}$ C. $\frac{5}{36}$ D. $\frac{3}{10}$ E. NOTA
5. A teacher is seating 4 troublesome students along a row of 10 seats. None of these students may be seated adjacent to each other. How many ways can the four students be seated?
A. 210 B. 420 C. 1344 D. 5040 E. NOTA

6. Let $k \in [4]$ be chosen uniformly. What is the expected value of the approximation of $\int_0^4 3x^2 dx$ given by a left-hand Riemann sum with k equal subintervals?
- A. $\frac{457}{18}$ B. $\frac{51}{2}$ C. $\frac{475}{18}$ D. 26 E. NOTA
7. Jordan is playing poker with Ben and needs to calculate the probability of certain hands to get an edge over him. A Flush is a hand of five cards in which all cards are the same suit. If the probability of drawing a flush in a five-card hand is $\frac{a}{\binom{52}{5}}$, what is the value of a ?
- A. 3861 B. 5148 C. 6395 D. 7682 E. NOTA
8. Let $f(x) = \sqrt{2 - x^2}$. Suppose k is uniformly selected from the domain of f . What is the probability that $f'(k) > f(k)$?
- A. $\frac{1}{8}$ B. $\frac{2 - \sqrt{2}}{4}$ C. $\frac{\sqrt{2}}{2}$ D. $\frac{2 + \sqrt{2}}{4}$ E. NOTA
9. At Frazlu High School there are 49 students taking Calculus, 25 students taking Computer Science, and 36 students taking Economics. Of those students, 10 are taking Calculus and Computer Science, 6 are taking Computer Science and Economics, and 15 are taking Economics and Calculus. Finally, there are 3 students who are taking all three courses. How many students in total are taking Calculus, Computer Science, or Economics?
- A. 55 B. 76 C. 79 D. 110 E. NOTA
10. Three dice are rolled. Given at least two of the numbers rolled are even, what is the probability that the sum of the numbers rolled is less than 8?
- A. $\frac{13}{108}$ B. $\frac{5}{36}$ C. $\frac{1}{6}$ D. $\frac{5}{27}$ E. NOTA

11. Define a **Latin square** to be an $n \times n$ matrix where each entry is an element of $[n]$ and each element of $[n]$ appears exactly once in each row and column. How many possible 3×3 Latin squares are there?
A. 12 B. 14 C. 16 D. 18 E. NOTA
12. A **coloring** is an assignment of colors to the vertices of a graph. Let $FANG$ be a square. If the available colors to use are black, pink, and white, how many colorings are there of $FANG$?
A. 9 B. 16 C. 27 D. 81 E. NOTA
13. A **proper coloring** is a coloring with the property that no two adjacent vertices have the same color. Let $FANG$ be a square. If the available colors to use are black, pink, and white, how many proper colorings are there of $FANG$?
A. 4 B. 6 C. 8 D. 10 E. NOTA
14. Henrik and Zhao play a game in which they take turns successively rolling dice. The first person who rolls a 6 wins. If Zhao goes first, what is the probability that he wins the game?
A. $\frac{6}{11}$ B. $\frac{7}{11}$ C. $\frac{8}{11}$ D. $\frac{9}{11}$ E. NOTA
15. What is the constant term in the expansion of $(-\frac{1}{x^2} + 2x)^9$?
A. -5376 B. -4608 C. 4608 D. 5376 E. NOTA

For questions 16-20: A **probability density function** $f_X(x)$ gives the relative probability of a value of a real-valued continuous random variable X . That is,

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

16. Suppose X is a real-valued continuous random variable with probability density function

$$f_X(x) = \begin{cases} a \cdot 3^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of a ?

- A. $\frac{1}{2} \ln 2$ B. $\frac{1}{2} \ln 3$ C. $\ln 2$ D. $\ln 3$ E. NOTA

17. What is $P(1 \leq X \leq 3)$?

- A. $\frac{2}{9}$ B. $\frac{7}{27}$ C. $\frac{8}{27}$ D. $\frac{1}{3}$ E. NOTA

18. What is the expected value of X ?

- A. $\frac{1}{9}$ B. $\frac{1}{3}$ C. $\ln 3$ D. $\sqrt{3}$ E. NOTA

19. What is the median of X ?

- A. $\frac{1}{\ln 3}$ B. $\frac{1}{\ln 2}$ C. $\frac{\ln 2}{\ln 3}$ D. 1 E. NOTA

20. Suppose Y is a continuous real-valued random variable with probability density function

$$f_Y(x) = \begin{cases} k \cdot (-x^4 + 5x^2 - 4) & a < |x| < b \\ 0 & \text{otherwise} \end{cases}$$

for some positive constants a, b, k . What is the value of $a^2 + b^2$ when k is minimized?

- A. 2 B. 4 C. 5 D. 9 E. NOTA

21. What is the expected number of digits which are equal to 1 in a randomly selected five-digit positive integer?
- A. $\frac{7}{15}$ B. $\frac{1}{2}$ C. $\frac{23}{45}$ D. $\frac{17}{30}$ E. NOTA
22. There is a meetup of 8 strangers. For each pair of strangers there is a $\frac{1}{2}$ chance that they will become friends. A friend triangle is defined as a group of 3 people in which all 3 people are friends. After the meetup, what is the expected number of friend triangles?
- A. 4 B. 5 C. 6 D. 7 E. NOTA
23. Let $k \in [0,1)$ be chosen uniformly at random. What is the expected value of the area bounded between $y = x - x^2$ and $y = kx$?
- A. $\frac{1}{24}$ B. $\frac{1}{12}$ C. $\frac{1}{8}$ D. $\frac{1}{6}$ E. NOTA
24. How many distinguishable permutations of the letters in ARKANSAS are there if the first and last letters must be different?
- A. 1920 B. 2400 C. 2880 D. 3360 E. NOTA
25. A lattice path between $(0,0)$ and $(5,5)$ consists of unit steps up and unit steps right. How many paths between $(0,0)$ and $(5,5)$ pass through $(3,3)$?
- A. 80 B. 120 C. 160 D. 200 E. NOTA

26. How many ways can we pick subsets $A, B \subseteq [4]$ such that $A \cap B \neq \emptyset$?
- A. 175 B. 202 C. 229 D. 256 E. NOTA
27. Four points A, B, C, D are chosen uniformly at random on a circle. What is the probability that \overline{AB} and \overline{CD} intersect?
- A. $\frac{1}{16}$ B. $\frac{1}{8}$ C. $\frac{1}{4}$ D. $\frac{1}{2}$ E. NOTA
28. A set of integers has the property that the sum or difference between any two members of the set is not divisible by 20. What is the maximum possible size of this set?
- A. 11 B. 12 C. 13 D. 14 E. NOTA
29. How many bijections $f: [6] \rightarrow [6]$ are there such that for exactly one $1 \leq i \leq 6, f(i) = i$?
- A. 256 B. 264 C. 272 D. 280 E. NOTA
30. Two real numbers x, y are chosen uniformly at randomly on the interval $(0,1)$. What is the probability that $\left\lceil \frac{x}{y} \right\rceil$ is an even number?
- A. $\frac{1}{2} \ln 2$ B. $\frac{1}{3} \ln 3$ C. $\frac{1}{5} \ln 5$ D. $\frac{1}{3}$ E. NOTA