1. D
2. B
3. B
4. C
5. C
6. C
7. A
8. C
9. A
10. B
11. C
12. D
13. B
14. A
15. A
16. D
17. B
18. D
19. A
20. B
21. A
22. B
23. C
24. A
25. B
26. B
27. A
28. D
29. D
30. E

On this test, answer choice (E) should be chosen when None Of The Answers listed are correct.

As a general hint, sometime the questions prior to trickier questions are a clue as to how to solve them.

(1) If $f(x) = \cos^4(x) - \sin^4(x)$, find $f'\left(\frac{\pi}{4}\right)$.

(A) $-\sqrt{3}$ (B) -1

(C) $-\sqrt{2}$ (D) -2 (E) NOTA

Solution:

$f(x) = \cos^4(x) - \sin^4(x) = (\cos^2(x) - \sin^2(x))(\cos^2(x) + \sin^2(x)) = \cos(2x)$. So $f'(x) = -2\sin(2x)$ and $f'\left(\frac{\pi}{4}\right) = -2\sin\left(2\frac{\pi}{4}\right) = -2$. $\boxed{D}$

(2) Alex's favorite Statistics function is $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$, where $\sigma > 0$ is constant. Find the positive value of x for which $f(x)$ has a point of inflection.

(A) 0 (B) σ

(C) $\sqrt{2}\sigma$ (D) σ^2 (E) NOTA

Solution:

$f'(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \cdot -\frac{x}{\sigma^2} = \frac{-1}{\sigma^2\sqrt{2\pi\sigma^2}} x e^{-\frac{x^2}{2\sigma^2}}$.

$f''(x) = \frac{-1}{\sigma^2\sqrt{2\pi\sigma^2}} x e^{-\frac{x^2}{2\sigma^2}}\left(-\frac{x}{\sigma^2}\right) + \frac{-1}{\sigma^2\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{\sigma^2\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x^2}{\sigma^2} - 1\right) \to x = \sigma$. $\boxed{B}$.

(3) Evaluate:

$$\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{3} + 5h\right) - \tan\left(\frac{\pi}{3} - 4h\right)}{3h} =$$

(A) 4 (B) 12

(C) 3 (D) $\frac{3}{4}$ (E) NOTA

Solution:

In the Limit definition of a derivative, the Δx in the denominator must match the difference in the points evaluated in the numerator. Therefore: $\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{3}+5h\right)-\tan\left(\frac{\pi}{3}-4h\right)}{3h} = 3 \cdot \lim_{h\to 0} \frac{\tan\left(\frac{\pi}{3}+5h\right)-\tan\left(\frac{\pi}{3}-4h\right)}{9h} = 3 \cdot \frac{d}{dx}[\tan(x)]\Big]_{x=\frac{\pi}{3}} = 3\sec^2\left(\frac{\pi}{3}\right) = 12$. $\boxed{B}$.

(4) Which of the following pairs of behaviors best describes $f(x) = x^5 - 7x^3 + 3x^2 - x + 2022$ at $x = 2$?

(A) Increasing, Concave Down (C) Increasing, Concave Up
(B) Decreasing, Concave Down (D) Decreasing, Concave Up (E) NOTA

Solution:

$f(x) = x^5 - 7x^3 + 3x^2 - x + 2 \to f'(x) = 5x^4 - 21x^2 + 6x - 1$ & $f''(x) = 20x^3 - 42x + 6$. So $f'(2) = 5 * 2^4 - 21 * 2^2 + 12 - 1 = 7 > 0$ & $f''(2) = 20 * 2^3 - 42 * 2 + 6 = 82 > 0$. $\boxed{C}$.

(5) If $f(x) = x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8$, find $\frac{d}{dx}[f^{-1}(x)]$ evaluated at $x = 20$

(A) 20 (B) 26
(C) $\frac{1}{26}$ (D) $\frac{1}{20}$ (E) NOTA

Solution:

In general, $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$. Clearly, $f(1) = 20$, so $f^{-1}(20) = 1$. Therefore $\frac{1}{f'(f^{-1}(20))} = \frac{1}{f'(1)}$. Finally $f(x) = x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8 \to f'(x) = 5x^4 + 4x^3 + 6x^2 + 6x + 5 \to f'(1) = 5 + 4 + 6 + 6 + 5 = 26$. So the answer is $\frac{1}{26}$. $\boxed{C}$.

(6) Find the largest value of $x > 0$ for which $\begin{bmatrix} x & 5x & 2 \\ -3 & \frac{1}{x} & 2 \\ 1 & 3 & 1 \end{bmatrix}$ does not have an inverse.

(A) $\frac{2}{19}$ (B) $\frac{17+\sqrt{137}}{38}$

(C) 1 (D) $\frac{17-\sqrt{137}}{38}$ (E) NOTA

Solution:

$\begin{vmatrix} x & 5x & 2 \\ -3 & \frac{1}{x} & 2 \\ 1 & 3 & 1 \end{vmatrix} = 1 + 10x - 18 - \frac{2}{x} - 6x + 15x = 19x - \frac{2}{x} - 17 = 0 \to 19x^2 - 17x - 2 = 0 \to x = \frac{17 \pm \sqrt{17^2 + 4(19)(2)}}{2(19)} = \frac{17 \pm 21}{38} = 1$. $\boxed{C}$.

(7) Find the minimal possible value of the determinant of $\begin{bmatrix} x & 5x & -2 \\ -3 & \frac{1}{x} & 2 \\ 1 & 3 & 1 \end{bmatrix}$ for $x > 0$. Note this is not the same matrix as the previous question.

(A) $19 + 2\sqrt{38}$ (B) $19 - 2\sqrt{38}$

(C) $\frac{\sqrt{38}}{2}$ (D) $\frac{\sqrt{38}}{19}$ (E) NOTA

Solution:

$\begin{vmatrix} x & 5x & -2 \\ -3 & \frac{1}{x} & 2 \\ 1 & 3 & 1 \end{vmatrix} = 1 + 10x + 18 + \frac{2}{x} - 6x + 15x = 19x + \frac{2}{x} + 19 = f(x) \rightarrow f'(x) = 0 = 19 - \frac{2}{x^2} \rightarrow$ $x = \sqrt{\frac{2}{19}} = \frac{\sqrt{38}}{19}$. The minimum possible value is therefore $19\left(\frac{\sqrt{38}}{19}\right) + 2\left(\frac{19}{\sqrt{38}}\right) + 19 = 19 + 2\sqrt{38}$ $\boxed{A}$.

(8) Evaluate

$$\int_{e^{e^e}}^{\infty} \frac{dx}{x(\ln(x))(\ln(\ln(x))\,(\ln(\ln(\ln(x))))^{2022}}$$

(A) $\frac{1}{2022}$ (B) $-\frac{1}{2022}$

(C) $\frac{1}{2021}$ (D) $-\frac{1}{2021}$ (E) NOTA

Solution:

Let $u = \ln(ln(ln(x))) \rightarrow du = \frac{1}{\ln(ln(x))\ln(x)x}dx$.

Therefore

$\int_{e^{e^e}}^{\infty} \frac{dx}{x(\ln(x))(\ln(ln(x))(\ln(ln(ln(x))))^{2022}} = \int_1^{\infty} u^{-2022}du = \left[-\frac{1}{2021}u^{-2021}\right]_1^{\infty} = \frac{1}{2021}$. $\boxed{C}$

(9) Find the maximum possible value of $\int_{-\pi}^{2\pi} f(x)\sin(x)\,dx$ if $|f(x)| \leq 2022$ everywhere.

(A) 12132 (B) 8088

(C) 4044 (D) 2022 (E) NOTA

Solution:

We want $f(x) = -2022$ when $\sin(x) < 0$ and $f(x) = 2022$ when $\sin(x) > 0$. If so the integral becomes $\int_{-\pi}^{2\pi} f(x)\sin(x)\,dx = \int_{-\pi}^{0} -2022\sin(x)\,dx + \int_0^{\pi} 2022\sin(x)\,dx + \int_{\pi}^{2\pi} -2022\sin(x)\,dx =$ $4044 + 4044 + 4044 = 12132$. $\boxed{A}$.

(10) Evaluate

$$\lim_{n\to\infty}\left(\frac{n^2 + 2000n + 4022}{n^2 + 2022n + 4044}\right)^n$$

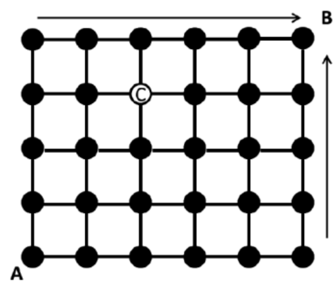
(A) e^{-11} (B) e^{-22}

(C) e^{-44} (D) e^{-88} (E) NOTA

Solution:

$\lim_{n\to\infty}\left(\frac{n^2+2000n+4022}{n^2+2022n+4044}\right)^n = \lim_{n\to\infty}\left(\frac{n^2+2022n+4044-22n-22}{n^2+2022n+4044}\right)^n = \lim_{n\to\infty}\left(1-22\left(\frac{n+1}{n^2+2022n+4044}\right)\right)^n =$ $\lim_{n\to\infty}\left(1-\frac{22}{n}\right)^n = e^{-22}$. $\boxed{B}$

(11) Consider the following grid of points:



Suppose that, starting at the point labeled A, Jae can go one step up or one step to the right at each move. This procedure is continued until Jae reaches the point labeled B. Given that Jae's path goes through the point C, what is the probability that Jae's first move was to the right, if every path is equally likely?

(A) $\frac{1}{10}$ (B) $\frac{1}{5}$

(C) $\frac{2}{5}$ (D) $\frac{3}{5}$ (E) NOTA

Solution:

It will take two steps right and three steps up to get to Point C, so the total number of paths from Point A to Point C is the number of arrangements on RRUUU which is $\frac{5!}{2!3!} = 10$. It will take one step up and three steps right to get to Point B, so the total number of paths from Point C to Point B is the number of arrangements of RRRU which is 4. Therefore there are (10)(4)=40 total paths from Point A to Point B through Point C. If the first step is to the right, then there will be only the number of arrangements of RUUU to get from A to C, which is four. Therefore there would be 4*4=16 total paths that go from A to B, go through C, and have a first step to the right. So the probability is 16/40=2/5. $\boxed{C}$.

(12) Urn A has 6 red balls and 6 purple balls. Urn B has 8 red balls and 7 purple balls. Iris rolls a fair 6-sided die. If the outcome is one, she selects a ball from Urn A. Otherwise, she selects a ball from Urn B. Suppose Iris selected a purple ball. What is the probability that the die roll was one?

(A) $\frac{1}{6}$ (B) $\frac{1}{2}$

(C) $\frac{7}{15}$ (D) $\frac{3}{17}$ (E) NOTA

Solution:

$P(Purple|One) = \frac{6}{12}$ & $P(Purple|{\sim}One) = \frac{7}{15} \rightarrow P(One|Purple) = \frac{P(One)P(Purple|One)}{P(One)P(Purple|One)+P({\sim}One)P(Purple|{\sim}One)} = \frac{\left(\frac{1}{6}\right)\left(\frac{6}{12}\right)}{\left(\frac{1}{6}\right)\left(\frac{6}{12}\right)+\left(\frac{5}{6}\right)\left(\frac{7}{15}\right)} = \frac{\frac{1}{12}}{\frac{1}{12}+\frac{7}{18}} = \frac{3}{3+14} = \frac{3}{17}$. $\boxed{D}$.

For the next three questions, use the following information:

A **probability density function** is a piecewise continuous non-negative function $f(x)$ with the property that the area between this function and the x-axis is one. The probability of random variables attaining ranges of values can be determined using probability density functions, such that the probability of $a < X < b$ is given by $\int_a^b f(x)dx$. The **expectation**, or mean value, of a random variable with probability density $f(x)$ is given by $\int_{\mathbb{R}} xf(x)dx$.

(13) Suppose $q(x) = \begin{cases} kx(2-x), 0 \leq x \leq 2 \\ 0, \text{Otherwise} \end{cases}$ is a probability density function. Find k.

(A) $\frac{3}{8}$ (B) $\frac{3}{4}$

(C) $\frac{4}{3}$ (D) $\frac{8}{3}$ (E) NOTA

Solution:

We need $k\int_0^2 2x - x^2\, dx = 1 \rightarrow k\left[x^2 - \frac{x^3}{3}\right]_0^2 = k\left(4 - \frac{8}{3}\right) = \frac{4}{3}k = 1 \rightarrow k = \frac{3}{4}$. $\boxed{B}$

(14) It is known that $r(x) = \begin{cases} \frac{1}{2}e^{-\sqrt{x}}, x \geq 0 \\ 0, \text{Otherwise} \end{cases}$ is a probability density function. Find the expectation of a random variable governed by this density function.

(A) 6 (B) 12

(C) 24 (D) 36 (E) NOTA

Solution:

The expectation is $\int_0^\infty \frac{1}{2}xe^{-\sqrt{x}}dx$. Using the substitution $u^2 = x \rightarrow 2u\, du = dx$, we get $\int_0^\infty u^3e^{-u}du$. We can use tabular method to complete this integration by parts to get the answer 6. $\boxed{A}$

(15) There are three unlabeled computers in the computer lab. One of these computers (call it Computer A) has a boot-up time (in minutes) that is a random variable governed by the probability density $q(x)$ above. The remaining two computers have boot-up times (in minutes) that are random variables governed by the probability density $r(x)$ above. Saathvik picks a

computer at random with equal probability, and finds that the boot-up time was less than one minute. What is the probability that he is working on Computer A?

(A) $\frac{e}{5e-8}$ (B) $\frac{e}{9e-16}$

(C) $\frac{1}{e}$ (D) $\frac{1}{3}$ (E) NOTA

Solution:

$P(X < 1|q(x)) = \int_0^1 \frac{3}{4}x(2-x)dx = \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}$. This is also obvious by symmetry.

$P(X < 1|r(x)) = \int_0^1 \frac{1}{2}e^{-\sqrt{x}}dx = \int_0^1 ue^{-u}du = [e^{-u}(-u-1)]_0^1 = 1 - \frac{2}{e}$.

Therefore $P(q(x)|X < 1) = \frac{P(q(x))P(X<1|q(x))}{P(q(x))P(X<1|q(x))+P(r(x))P(X<1|r(x))} = \frac{\frac{1}{3}\left(\frac{1}{2}\right)}{\frac{1}{3}\left(\frac{1}{2}\right)+\left(\frac{2}{3}\right)\left(1-\frac{2}{e}\right)} = \frac{1}{1+4-\frac{8}{e}} = \frac{1}{\frac{se-8}{e}} = \frac{e}{5e-8}$. $\boxed{A}$

(16) If $Re(e^{i\alpha}) = \frac{5}{13}$ and $Im(e^{i\beta}) = \frac{3}{5}$, and both $-\frac{\pi}{2} < \alpha < 0$ and $\frac{\pi}{2} < \beta < \pi$ and then find $\tan(\alpha+\beta)$.

(A) $-\frac{33}{56}$ (B) $\frac{33}{56}$

(C) $-\frac{63}{16}$ (D) $\frac{63}{16}$ (E) NOTA

Solution:

$Re(e^{i\alpha}) = \cos(\alpha) = \frac{5}{13} \to \sin(\alpha) = -\frac{12}{13} \to \tan(\alpha) = -\frac{12}{5}$.

$Im(e^{i\beta}) = \sin(\beta) = \frac{3}{5} \to \cos(\beta) = -\frac{4}{5} \to \tan(\beta) = -\frac{3}{4}$.

So $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} = \frac{-\frac{12}{5}-\frac{3}{4}}{1-\left(-\frac{12}{5}\right)\left(-\frac{3}{4}\right)} = \frac{63}{16}$. $\boxed{D}$

(17) Compute

$$\sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n!}$$

(A) $\sqrt{e}\cos\left(\frac{\sqrt{3}}{2}\right)$ (B) $\sqrt{e}\sin\left(\frac{\sqrt{3}}{2}\right)$

(C) $\frac{\sqrt{e}}{2}$ (D) $\frac{\pi\sqrt{e}}{3}$ (E) NOTA

Solution:

In general, $\sum_{n=0}^{\infty}\frac{\sin(nx)}{n!} = Im\left(\sum_{n=0}^{\infty}\frac{e^{inx}}{n!}\right) = Im\left(\sum_{n=0}^{\infty}\frac{\left(e^{ix}\right)^n}{n!}\right) = Im\left(e^{e^{ix}}\right) = Im\left(e^{\cos(x)+i\sin(x)}\right) = Im\left(e^{\cos(x)}\right)\left(\cos(sin(x)) + i\sin\left(\sin(x)\right)\right) = e^{\cos(x)}\sin(\sin(x))$. Letting $x = \frac{\pi}{3} \to \cos(x) = \frac{1}{2}$ & $\sin(x) = \frac{\sqrt{3}}{2} \to$ The final answer is $e^{\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right) = \sqrt{e}\sin\left(\frac{\sqrt{3}}{2}\right)$. $\boxed{B}$

(18) Consider the parabola $y = ax^2 + 2022x + 2022$. There exists exactly one circle which has its center on the x-axis and is tangent to the parabola at exactly two points. It turns out one of the tangent points is $(0, 2022)$. Determine the value of a.

(A) $\frac{1}{2022}$ (B) $-\frac{1}{2022}$

(C) $\frac{1}{4044}$ (D) $-\frac{1}{4044}$ (E) NOTA

Solution:

By symmetry, since the vertex of this parabola occurs at $x = -\frac{1011}{a}$, then the center of the circle must be at $\left(-\frac{1011}{a}, 0\right)$. Now the slope of the parabola at $(0, 2022)$ is 2022, so the line connecting $\left(-\frac{1011}{a}, 0\right)$ to $(0, 2022)$ must have a slope of $-\frac{1}{2022}$. Thus $\frac{2022}{\left(\frac{1011}{a}\right)} = -\frac{1}{2022} \to 2a = -\frac{1}{2022} \to a = -\frac{1}{4044}$. $\boxed{D}$

(19) Define the functions $f_0, f_1, f_2, f_3, \dots, f_n, \dots$ to satisfy the relations $f_0 = \frac{1}{2}$ and $f_n' = f_{n-1}$. Furthermore, assume for all $n > 0$ that $f_n(0) = 0$. Evaluate

$$\sum_{n=0}^{\infty} f_n(\ln(2022))$$

(A) 1011 (B) 2022

(C) 4044 (D) 8088 (E) NOTA

Solution:

$$f_1' = f_0 = \frac{1}{2} \to f_1 = \frac{1}{2}x$$

$$f_2' = \frac{1}{2}x \to f_2 = \frac{1}{2}\frac{x^2}{2}$$

$$f_3' = \frac{1}{2}\frac{x^2}{2} \to f_3 = \frac{1}{2}\frac{x^3}{3\cdot 2}$$

And in general $f_3 = \frac{1}{2}\frac{x^n}{n!} \to \sum_{n=0}^{\infty} f_n(\mathrm{x}) = \sum_{n=0}^{\infty} \frac{1}{2}\frac{x^n}{n!} = \frac{1}{2}e^x \to$ The answer is $\frac{1}{2}e^{\ln(2022)} = 1011.$ $\boxed{A}$

(20) Let $f(x)$ be a continuous, differentiable function at $x = 0$ with $f(0) = 0$ and $f'(0) = 2022$. Evaluate

$$\lim_{x\to 0} \frac{f(x)}{x}$$

(A) 1011 (B) 2022

(C) 4044 (D) 8088 (E) NOTA

Solution:

By l'Hopital the answer is just $f'(0) = 2022$. $\boxed{B}$.

(21) Evaluate

$$\lim_{x\to 0} \frac{\sin^2(4044x)\tan^3(1011x)}{(\ln(2022\mathrm{x}+1))^5}$$

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$

(C) $\frac{1}{8}$ (D) $\frac{1}{16}$ (E) NOTA

Solution:

From the previous question it is easy to see that for any function with $f(0) = 0$ and $f'(0) = a$, $\lim_{x\to 0} \frac{f(x)}{x} = a$. We can therefore use manipulation and limit rules to write the limit as

$$\lim_{x\to 0} \frac{\sin^2(4044x)\tan^3(1011x)}{(\ln(2022\mathrm{x}+1))^5} = \frac{\left(\lim_{x\to 0}\frac{\sin(4044x)}{x}\right)^2\left(\lim_{x\to 0}\frac{\tan(1011x)}{x}\right)^3}{\left(\lim_{x\to 0}\frac{\ln(2022\mathrm{x}+1)}{x}\right)^5}$$

Each of these limits is of the form above, and so the answer is

$$\frac{\left(\lim_{x\to 0}\frac{\sin(4044x)}{x}\right)^2\left(\lim_{x\to 0}\frac{\tan(1011x)}{x}\right)^3}{\left(\lim_{x\to 0}\frac{\ln(2022\mathrm{x}+1)}{x}\right)^5} = \frac{4044^2 1011^3}{2022^5} = \frac{2^4 1011^5}{2^5 1011^5} = \frac{1}{2}$$

$\boxed{A}$

(22) Let $\mathcal{L}_1$ be the line in space with directional vector $< 1, -2, 2 >$ going through the point $(0, 1, 2)$. Let $\mathcal{L}_2$ be the line in space with directional vector $< 3, -1, -4 >$ going through the point $(3, -1, 2)$. Find the minimum distance between these lines.

(A) 0 (B) $\frac{2}{3}$

(C) $\frac{10}{3}$ (D) $\frac{2}{5}$ (E) NOTA

Solution:

The distance between skew lines is the length of the vector between any point on one line to the other projected onto a vector perpendicular to both lines. So, we define $\boldsymbol{n} =< 1,-2,2 > \times < 3,-1,-4 >= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 2 \\ 3 & -1 & -4 \end{vmatrix} =< 10,10,5 >$. Then we will project $\boldsymbol{v} =< 3-0,-1-1,2-2 >=< 3,-2,0 >$ onto $\boldsymbol{n}$ and take its magnitude. This is equivalent to $\frac{|<3,-2,0>\cdot<10,10,5>|}{\|<10,10,5>\|} = \frac{10}{15} = \frac{2}{3}$. $\boxed{B}$

(23) Consider a line with direction vector $< 2,-2,1 >$ going through the point $(2,3,4)$. Which of the following expressions represents the infinite cylinder in space that is the locus of all points a distance of 5 away from this line?

(A) $(y+2z-11)^2 + (x-2z+6)^2 + (x+2y-10)^2 = 25$

(B) $(x-2)^2 + (y-3)^2 + (z-4)^2 = 25$

(C) $(y+2z-11)^2 + (x-2z+6)^2 + (x+2y-10)^2 = 225$

(D) $(y+2z+11)^2 + (x-2z-6)^2 + (x+2y+10)^2 = 25$

(E) NOTA

Solution:

The distance from a point to a line in 3-space is given by the equation $d = \frac{\|\overrightarrow{PQ}\times\vec{v}\|}{\|\vec{v}\|}$. Here $\overrightarrow{PQ}$ is a vector connecting the point of interest to a point on the line and $\vec{v}$ is the direction vector of the line. Let a point on the cylinder by denoted (x,y,z). Therefore $\overrightarrow{PQ} =< x-2, y-3, z-4 >$. Thus we have $\overrightarrow{PQ}\times\vec{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x-2 & y-3 & z-4 \\ 2 & -2 & 1 \end{vmatrix} =< y-3+2z-8, 2z-8-x+2, -x+4-2y+6 >=< y+2z-11, 2z-x-6, -x-2y+10 >$. And so $d^2 = 25 = \frac{\|\overrightarrow{PQ}\times\vec{v}\|^2}{\|\vec{v}\|^2} = \frac{(y+2z-11)^2+(x-2z+6)^2+(x+2y-10)^2}{9} \rightarrow (y+2z-11)^2 + (x-2z+6)^2 + (x+2y-10)^2 = 225$. $\boxed{C}$

(24) Let $f(x)$ be a continuous, differentiable function satisfying $x = f(x)e^{f(x)}$. Calculate $\int_0^e f(x)dx$.

(A) $e-1$ (B) e

(C) e^{e-1} (D) e^e (E) NOTA

Solution:

Start with the substitution $u = f(x) \to du = f'(x)dx$. Note that $1 = f'(x)e^{f(x)} + f(x)f'(x)e^{f(x)}$ so that $f'(x) = \frac{1}{e^{f(x)}+f(x)e^{f(x)}} = \frac{1}{e^{f(x)}+x} = \frac{1}{\frac{x}{f(x)}+x} = \frac{f(x)}{x(f(x)+1)}$. Therefore $\int f(x)dx = \int u \cdot \frac{du}{\frac{u}{x(u+1)}} =$ $\int x(u+1)du = \int ue^u(u+1)du = e^u(u^2-u+1) = e^{f(x)}(f(x)^2 - f(x)+1)$. Completing the definite integral, note that $0 = f(0)e^{f(0)} \to f(0) = 0$ and $e = f(e)e^{f(e)} \to f(e) = 1$. So the answer is

$e^1(1^2-1+1) - e^0(0^2-0+1) = e-1$. $\boxed{A}$

(25) Let $f(x)$ be an odd, continuously differentiable function with $f(\pi) = 2022$.

Evaluate

$$\int_{-\pi}^{\pi} \frac{f'(x)}{1-\sin(x)+\sqrt{1+\sin^2(x)}}dx$$

(A) 1011 (B) 2022

(C) 4044 (D) 8088 (E) NOTA

Solution:

Use the bounds trick substitution $u = -x \to du = -dx$. Then

$\int_{-\pi}^{\pi} \frac{f'(x)}{1-\sin(x)+\sqrt{1+\sin^2(x)}}dx = \int_{\pi}^{-\pi} -\frac{f'(-u)}{1-\sin(-u)+\sqrt{1+\sin^2(-u)}}du = \int_{-\pi}^{\pi} \frac{f'(u)}{1+\sin(u)+\sqrt{1+\sin^2(u)}}dx$ since $f'(u)$ is even if $f(x)$ is odd.

Therefore

$2\int_{-\pi}^{\pi} \frac{f'(x)}{1-\sin(x)+\sqrt{1+\sin^2(x)}}dx = \int_{-\pi}^{\pi} \frac{f'(x)}{1-\sin(x)+\sqrt{1+\sin^2(x)}}dx + \int_{-\pi}^{\pi} \frac{f'(x)}{1+\sin(x)+\sqrt{1+\sin^2(x)}}dx =$

$\int_{-\pi}^{\pi} f'(x)\left(\frac{1}{1-\sin(x)+\sqrt{1+\sin^2(x)}} + \frac{1}{1+\sin(x)+\sqrt{1+\sin^2(x)}}\right)dx =$

$\int_{-\pi}^{\pi} f'(x)\left(\frac{1+\sin(x)+\sqrt{1+\sin^2(x)}+1-\sin(x)+\sqrt{1+\sin^2(x)}}{\left(1-\sin(x)+\sqrt{1+\sin^2(x)}\right)\left(1+\sin(x)+\sqrt{1+\sin^2(x)}\right)}\right)dx$

$= \int_{-\pi}^{\pi} f'(x)\left(\frac{2+2\sqrt{1+\sin^2(x)}}{1+\sin(x)+\sqrt{1+\sin^2(x)}-\sin(x)-\sin^2(x)-\sin(x)\sqrt{1+\sin^2(x)}+\sqrt{1+\sin^2(x)}+\sin(x)\sqrt{1+\sin^2(x)}+1+\sin^2(x)}\right)dx$

$= \int_{-\pi}^{\pi} f'(x)(1)dx = [f(x)]_{-\pi}^{\pi} = f(\pi) - f(-\pi) = 4044.$

So $\int_{-\pi}^{\pi} \frac{f'(x)}{1-\sin(x)+\sqrt{1+\sin^2(x)}}dx = \frac{4044}{2} = 2022$ $\boxed{B}$.

(26) Find the area enclosed by the locus of all points that are exactly $\frac{1}{n-1}$ as far away from the point $(1, 2)$ as they are from the line $y = -2x + 24$ for $n > 2$.

(A) $\frac{40(n-1)}{(n(n-2))^{\frac{3}{2}}}\pi$ (B) $\frac{80(n-1)}{(n(n-2))^{\frac{3}{2}}}\pi$

(C) $\frac{40\sqrt{n(n-2)}}{n-1}\pi$ (D) $\frac{80\sqrt{n(n-2)}}{n-1}\pi$ (E) NOTA

Solution:

This describes an ellipse with eccentricity $\frac{1}{n-1}$ with focus $(1{,}2)$ and directrix $y = -2x + 24$. The first thing to do is figure out the location of one of the vertices. To do this, we note that the line perpendicular to the directrix containing the focus is $y - 2 = \frac{1}{2}(x-1) \to y = \frac{1}{2}x + \frac{3}{2}$. This intersects the directrix when $\frac{1}{2}x + \frac{3}{2} = -2x + 24 \to \frac{5}{2}x = \frac{55}{2} \to x = 9 \to y = 6$. The point that is $\frac{1}{n-1}$ the distance from $(1{,}2)$ to $(9{,}6)$ is $\left(\frac{(n-1)(1)+9}{n}, \frac{(n-1)(2)+6}{n}\right) = \left(1 + \frac{8}{n}, 2 + \frac{4}{n}\right)$. This is the location of one of the vertices. The distance from $(1{,}2)$ to $\left(1 + \frac{8}{n}, 2 + \frac{4}{n}\right)$ is $\sqrt{\left(\frac{8}{n}\right)^2 + \left(\frac{4}{n}\right)^2} = \frac{4\sqrt{5}}{n} = a - c$ where a is the length of the semi-major axis and $c = \sqrt{a^2 - b^2}$ is the focal distance, with b the length of the semi-minor axis. We also know that $e = \frac{c}{a} = \frac{1}{n-1}$. Therefore $c = \frac{1}{n-1}a \to \frac{4\sqrt{5}}{n} = a - \frac{1}{n-1}a = \frac{n-2}{n-1}a \to a = \frac{4(n-1)\sqrt{5}}{n(n-2)}$ and $c^2 = a^2 - b^2 \to \frac{1}{(n-1)^2}a^2 = a^2 - b^2 \to b^2 = \frac{(n-1)^2-1}{(n-1)^2}a^2 \to b = \frac{\sqrt{n(n-2)}}{n-1}a = \frac{\sqrt{n(n-2)}}{n-1} \cdot \frac{4(n-1)\sqrt{5}}{n(n-2)} = \frac{4\sqrt{5}}{\sqrt{n(n-2)}}$. Thus the area is $\pi ab = \pi\frac{4(n-1)\sqrt{5}}{n(n-2)}\frac{4\sqrt{5}}{\sqrt{n(n-2)}} = \frac{80(n-1)}{(n(n-2))^{\frac{3}{2}}}\pi$. $\boxed{B}$

(27) Find the rate of change of the area enclosed by the locus of all points that are exactly $\frac{1}{n-1}$ as far away from the point $(1, 2)$ as they are from the line $y = -2x + 24$ for if n is increasing 48 units/hour and $n = 6$.

(A) $-\frac{85\sqrt{24}\pi}{6}$ (B) $-\frac{85\sqrt{24}\pi}{12}$

(C) $\frac{85\sqrt{24}\pi}{6}$ (D) $\frac{85\sqrt{24}\pi}{12}$ (E) NOTA

Solution:

From the previous question, $A(n) = \frac{80(n-1)}{(n(n-2))^{\frac{3}{2}}}\pi \to \frac{dA}{dt} = \frac{(n(n-2))^{\frac{3}{2}}(80) - 80(n-1)\left(\frac{3}{2}\right)(n(n-2))^{\frac{1}{2}}(2n-2)}{(n(n-2))^3}\frac{dn}{dt}\pi = \frac{80(\sqrt{24})^3 - 80(3)(5^2)\sqrt{24}}{24^3}(48)\pi = \frac{1920-6000}{288}\sqrt{24}\pi = -\frac{85\sqrt{24}\pi}{6}$. $\boxed{A}$

(28) If $t = \tan\left(\frac{\theta}{2}\right)$, and $0 < \theta < \frac{\pi}{2}$, then express $\sin(\theta) + \cos(\theta)$ in terms of t.

(A) $\frac{1+t}{\sqrt{1+t}}$ (B) $\frac{1+t}{\sqrt{1+t^2}}$

(C) $\frac{(1+t)^2}{1+t^2}$ (D) $\frac{1+2t-t^2}{1+t^2}$ (E) NOTA

Solution:

$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{t}{1} = \frac{\frac{t}{\sqrt{1+t^2}}}{\frac{1}{\sqrt{1+t^2}}} \to \sin\left(\frac{\theta}{2}\right) = \frac{t}{\sqrt{1+t^2}}$ & $\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+t^2}}$. So $\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{2t}{1+t^2}$ and $\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$. Add them together to get $\boxed{D}$.

(29) Evaluate:

$$\int_0^{\pi} \ln(1 - 4044\cos(x) + 2022^2)\, dx$$

(A) $\pi\ln(4044)$ (B) $2\pi\ln(4044)$

(C) $\pi\ln(2022)$ (D) $2\pi\ln(2022)$ (E) NOTA

Solution:

The form and use of 2022 indicates one should consider $\int_0^{\pi} \ln(1 - 2a\cos(x) + a^2)\, dx$ and use Feynman's integration technique: $I(a) = \int_0^{\pi} \ln(1 - 2a\cos(x) + a^2)\, dx \to I'(a) = \int_0^{\pi} \frac{2a - 2\cos(x)}{1 - 2a\cos(x) + a^2} dx$. We use the substitution $t = \tan\left(\frac{x}{2}\right) \to \cos(\theta) = \frac{1-t^2}{1+t^2}$ & $dx = \frac{2}{1+t^2} dt$ for integrals of this type: $I'(a) = \int_0^{\pi} \frac{2a - 2\cos(x)}{1 - 2a\cos(x) + a^2} dx = \int_0^{\infty} \frac{2a - 2\frac{1-t^2}{1+t^2}}{1 - 2a\frac{1-t^2}{1+t^2} + a^2} \frac{2}{1+t^2} dt = \frac{2}{a}\int_0^{\infty} \frac{(a+1)(a-1)}{(a+1)^2t^2 + (a-1)^2} dt + \frac{2}{a}\int_0^{\infty} \frac{1}{1+t^2} dt = \frac{2}{a}\left[\arctan\left(\frac{(a+1)t}{a-1}\right) + \arctan(t)\right]_0^{\infty} = \frac{2\pi}{a} = I'(a) \to I(a) = 2\pi\ln(a)$. So the answer is $2\pi\ln(2022)$, $\boxed{D}$.

(30) You made it to the end of the test! Congratulations! Now, evaluate

$$\lim_{x\to 4} \frac{x + 4}{x^2 + \sqrt{x - 4}}$$

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4 (E) NOTA

Solution:

Since $\sqrt{x - 4}$ is not defined for values less than 4, and the limit is two-sided, this limit does not exist. $\boxed{E}$