1. D
2. C
3. B
4. C
5. D
6. A
7. D
8. C
9. A
10. D
11. A
12. B
13. C
14. B
15. A
16. C
17. E
18. C
19. B
20. B
21. D
22. D
23. C
24. A
25. D
26. B
27. E
28. A
29. B
30. C

1. D $1+2+\cdots+2022=1011\cdot 2023\equiv 3\ (mod 10)$.
2. C If $a_n=a+(n-1)d$, then $\frac{a_5}{a_3}=\frac{a+4d}{a+2d}=2$. This implies that $a=0$. So $\frac{a_4}{a_2}=\frac{3d}{d}=3$.
3. B $$\frac{2022+1}{7}=289$$
4. C If we let $a_1=m$, then $a_1+a_2+\cdots+a_n=m+(m+1)+\cdots+(m+n-1)=\frac{n(2m+n-1)}{2}=2022$. Then we have $n(2m+n-1)=4044=2^2\cdot 3\ \cdot 337$. The largest possible value for $n=12$.
5. D If we let $S=1+2x+3x^2+\cdots$, then $xS=x+2x^2+3x^3+\cdots$. Subtracting xS from S, we obtain $(1-x)S=1+x+x^2+\cdots=\frac{1}{1-x}$. Thus, $S=\frac{1}{(1-x)^2}$. Solving $\frac{1}{(1-x)^2}=25$, we get $x=\frac{4}{5}$.
6. A Dividing both sides of $na_{n+1}=2(n+1)a_n$ by $n(n+1)$, we get $\frac{a_{n+1}}{n+1}=2\frac{a_n}{n}$. This implies that $\frac{a_n}{n}$ is a geometric sequence with the initial term $\frac{a_1}{1}=1$. Therefore, $\frac{a_n}{n}=2^{n-1}$, or $a_n=n2^{n-1}$. Since $a_{101}=101\cdot 2^{100}\equiv 1\cdot 6\equiv 6(mod 10)$, the units digit value is 6.
7. D Let $13m-39, 13m-26, 13m-13, 13m, 13m+13, 13m+26, 13m+39$ be 7 consecutive positive multiples of 13. Then the sum is $91m$ where $m\ \geq 4$ and $91m\ \leq 2022$. There are 19 integers m satisfying the conditions.
8. C If $a_n=ar^{n-1}$, then $a_3=ar^2=12$ and $a_9=ar^8=75$. Then $a_6=ar^5=\sqrt{a_3a_9}=\sqrt{12\cdot 75}=30$.
9. A $a_{19}=(a_1+a_2+\cdots+a_{19})-(a_1+a_2+\cdots+a_{18})=(19!+20!)-(18!+19!)=20!-18!\equiv 0+1\ \equiv\ 1\ (mod 19)$.
10. D Taking log_2 on both sides of the recurrence relation $b_{n+2}=b_{n+1}b_n$, we get $log_2 b_{n+2}=log_2 b_{n+1}+log_2 b_n$. Then $log_2 b_n$ is the Fibonacci sequence. Thus $log_2 b_{11}=89$ or $b_{11}=2^{89}$. Hence, the units digit of b_{11} is 2.
11. A Note that $a_n\cdot a_{n+1}=(n^2+1)((n+1)^2+1)=(n(n+1)+1)^2+1=a_{n^2+n+1}$. Therefore, $a_{11}\cdot a_{12}\cdot a_{13}\cdot a_{14}=a_{133}\cdot a_{183}$. So $p+q=133+183=316$.
12. B Since the perimeter of the right triangle is 60, we can let the three sides be $20-d, 20, 20+d$. Then it follows from the Pythagorean theorem that $(20-d)^2+20^2=(20+d)^2$. Solving the equation, we get $d=5$. Therefore, the hypotenuse is of length 25.
13. C Since a_n is an arithmetic sequence, $a_1+a_2+\cdots+a_99=99a_{50}=9999$. Hence, $a_{50}=101$.
14. B Note that $S=(40-38)(40+38)+(39-37)(39+37)+\cdots+(4-2)(4+2)+(3-1)(3+1)=2(40+39+38+\cdots+2+1)=40\cdot 41=1640$. Thus, the hundreds digit of S is 6.
15. A Note that $a_1+a_2+\cdots+a_{20}=6(a_1+a_2+\cdots+a_{10})$. If we let $a_n=a+(n-1)d$, then we get $\frac{20(2a+19d)}{2}=6\frac{10(2a+9d)}{2}$. Simplifying the expression we get $a=-2d$. Thus, $\frac{a_1}{a_2}=\frac{a}{a+d}=2$.
16. C Let $2, 2r, and\ 2r^2$ be the three side lengths of the right triangle. Then by Pythagorean theorem we get $2^2+(2r)^2=(2r^2)^2$ or $r^4-r^2-1=0$. Solving the

equation for r^2, we obtain $r^2 = \frac{1+\sqrt{5}}{2}$. Therefore, the hypotenuse is of length $2r^2 = 1+\sqrt{5}$.

17. E $i + i^2 + \cdots + i^{2022} = i + i^2 = -1 + i$ (NOTA)

18. C $a_4 = (a_1)^8$, call $a_1 = e^{i\theta}$, we have $e^{i\theta} = e^{8i\theta}$, thus $8\theta = \theta + 2\pi k$. So the smallest positive $\theta = \frac{2\pi}{7}$, and $2 + 7 = 9$.

19. B Let $a-2d, a-d, a, a+d, a+2d$ where $d > 0$ be the five interior angles of a convex pentagon. Then $5a = 540$, so $a = 108$. Since it is a convex pentagon, we must have $108-2d > 0$ and $108+2d < 180$. Thus, $d \leq 35$, so the smallest possible value of the smallest interior angle is $108 - 2 \cdot 35 = 38$.

20. B Note that the sixth term, a_6, of an arithmetic sequence is the average of the first term, a_1, and the eleventh term, a_{11}. Hence, $a_6 = \frac{a_1 + a_{11}}{2} = \frac{\frac{3}{7}+\frac{2}{3}}{2} = \frac{23}{42}$. Therefore, $p + q = 23 + 42 = 65$.

21. D Let $S = L_0 + \frac{L_1}{2} + \frac{L_2}{4} + \frac{L_3}{8} + \cdots$. Then $2S = 2 \cdot 2 + L_1 + \frac{L_2}{2} + \frac{L_3}{4} + \cdots$. By adding the two expressions, we obtain $3S = 4 + (L_0 + L_1) + \frac{L_1+L_2}{2} + \frac{L_2+L_3}{4} + \cdots = 4 + \mathrm{L}_2 + \frac{\mathrm{L}_3}{2} + \frac{\mathrm{L}_4}{4} + \frac{\mathrm{L}_5}{8} \cdots = 4 + 4\left(\frac{\mathrm{L}_2}{4} + \frac{\mathrm{L}_3}{8} + \cdots\right) = 4 + 4\left(\mathrm{S} - \mathrm{L}_0 - \frac{\mathrm{L}_1}{2}\right) = 4 + 4\left(\mathrm{S} - \frac{5}{2}\right)$. Solving it for, we get $S = 6$.

22. D Let the three numbers be a, ar, and ar^2. Then the sum of the three numbers must be 365, i.e. $a + ar + ar^2 = 365$. Thus, $a(1 + r + r^2) = 365 = 5 \cdot 73$. Then, $a = 5$ and $1 + r + r^2 = 73$. Solving the quadratic for r, we get $r = 8$. So, there are $ar^2 = 5 \cdot 8^2 = 320$ days from Clair's birthday to Alicia's birthday.

23. C Note that $a_n = S_n - S_{n-1} = n^3 - (n-1)^3 = 3n^2 - 3n + 1$. Thus, $a_{2022} = 3 \cdot 2022^2 - 3 \cdot 2022 + 1 \equiv 3 \cdot 4 - 3 \cdot 2 + 1 \equiv 7 \ (mod\ 10)$.

24. A Note that $\frac{2n}{n^4+n^2+1} = \frac{2n}{(n^2-n+1)(n^2+n+1)} = \frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}$. Then $\sum_{n=1}^{\infty} \frac{2n}{n^4+n^2+1} = \left(\frac{1}{1^2-1+1} - \frac{1}{1^2+1+1}\right) + \left(\frac{1}{2^2-2+1} - \frac{1}{2^2+2+1}\right) + \cdots = 1$by telescoping series cancellation.

25. D Note that $A^3 + B^3 + C^3 = 3ABC$ when $A + B + C = 0$. Since $F_{13} + (-F_{12}) + (-F_{11}) = 0$, we obtain $F_{13}^3 - F_{12}^3 - F_{11}^3 = 3F_{13}F_{12}F_{11} = 3 \cdot 233 \cdot 144 \cdot 89 = 2^4 \cdot 3^3 \cdot 89 \cdot 233$. Thus, there are $(4+1) \cdot (3+1) \cdot (1+1) \cdot (1+1) = 80$ divisors.

26. B Listing a few early terms $1,2,3,4,2,2,-1,-2,-3,-4,-2,-2,1,2,3,4,\cdots$ we find that the sequence repeats with the period length 12. In other words, $a_{n+12} = a_n$ for all n. Since $2022 \equiv 6\ (mod\ 12)$, $a_{2022} = a_6 = 2$.

27. E Note that $f(1-x) = \frac{2}{4^{1-x}+2} = \frac{4^x}{4^x+2}. Therefore,\ f(x) + f(1-x) = 1$. It follows from this relationship that $f\left(\frac{1}{2022}\right) + f\left(\frac{2}{2022}\right) + f\left(\frac{3}{2022}\right) + \cdots + f\left(\frac{2021}{2022}\right) = 1010.5$.

28. A The sum can be written as $S = 101^3 + 100^3 + \cdots + 2^3 + 1^3 - 2(100^3 + 98^3 + \cdots + 2^3) = \left(\frac{101 \cdot 102}{2}\right)^2 - 16\left(\frac{50 \cdot 51}{2}\right)^2 \equiv 51^2 - 0 \equiv 1(mod 100)$. Hence, the tens digit of S is 0.

29. B Note that $P_n = \frac{3n^2-n}{2}$. Thus, $P_{20} = 590$.

30. C Let $P_n = \frac{3n^2-n}{2} = m^2$ for some positive integer m. Then the equation is equivalent to $3\left(n-\frac{1}{6}\right)^2 - 2m^2 = \frac{1}{12}$ or $(6n-1)^2 - 24m^2 = 1$. This is a Pell's equation whose smallest solution is $6n-1=5$ and $m=1$. Since we look for the smallest solution $n>1$, we compute $\left(5+\sqrt{24}\right)^k$ for $k=2,3,\cdots,$ until an integer value n is determined. The smallest case comes when $k=3$ where $\left(5+\sqrt{24}\right)^3 = 485 + 99\sqrt{24}$. Setting $6n-1=485$ and $m=99$, we obtain the desired answer $n=81$.