Mu

Area and Volume

Test #631

Directions:

1. Fill out the top left section of the scantron. Do not abbreviate your school name.

2. In the Student ID Number grid, write your 9-digit ID# and bubble.

3. In the Test Code grid, write the 3-digit test# on this test cover and bubble.

4. Scoring for this test is 5 times the number correct plus the number omitted.

5. TURN OFF ALL CELL PHONES.

6. No calculators may be used on this test.

7. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

8. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

9. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

10. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

11. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

12. Unless a question asks for an approximation or a rounded answer, give the exact answer.

All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above."

~~~~~~~~ Good luck, and have fun! ~~~~~~~~

1. A regular hexagon with side length 6 is inscribed in a circle. If the area inside the circle but outside the hexagon is equal to $6\left(A\pi - B\sqrt{C}\right)$ for squarefree C, find $A+B+C$.

   A. 18
   B. 33
   C. 69
   D. 93
   E. NOTA

2. Trevor clumsily trips over their own shoelaces while jumping over a convex quadrilateral with vertices at $(-3,8)$, $(2,-1)$, $(4,5)$, and $(0,9)$. While laughing, Sanjoy correctly calculates the area of this quadrilateral. What value does he obtain?

   A. 28
   B. 30.5
   C. 31.5
   D. 32
   E. NOTA

3. Nico attaches solid unit cubes to the outside centers of each face of a solid cube with side length 3. To the nearest percent, by how much has Nico increased the surface area of the solid?

   A. 22%
   B. 44%
   C. 55%
   D. 56%
   E. NOTA

4. Let A be the area of the region between the curves $y = ax^2 - 1$ and $y = 1 - 2ax^2$ for some real constant a. If a is increasing at a rate of 2 units per second, find the instantaneous rate of change of the area of A when $a = 6$.

   A. $-\frac{1}{27}$
   B. $-\frac{2}{27}$
   C. $-\frac{4}{27}$
   D. $-\frac{8}{27}$
   E. NOTA

5. A full cylindrical water tank of radius 3 and height 8 drains into an empty inverted conical tank with radius 8 and height 12. The height of the water in the cylindrical tank is decreasing at a constant rate of 1 per second. Find the rate of increase of the water height in the conical tank per second when the cylindrical tank is half full. (You may assume that water is transferred instantaneously and that there is no leakage.)

   A. $\frac{\sqrt[3]{3}}{3}$
   B. $\frac{1}{2}$
   C. $\frac{\sqrt[3]{9}}{4}$
   D. $\frac{\sqrt[3]{2}}{2}$
   E. NOTA

6. Corey has an infinite number of cubic dice for playing Dungeons and Dragons. His largest die has edge length 1 inch, and for each die in his collection, there are two dice with edge length 40% of it. Let the total volume of dice Corey has in cubic inches equal $\frac{D}{M}$. Find $D + M$.

A. 6
C. 234
E. NOTA
B. 186
D. 242

7. If $\int_0^1 \frac{x^8+1}{x^2+1}\, dx = \frac{\pi}{A} - \frac{B}{C}$, find the remainder when $A + B + C$ is divided by 9.

A. 0
C. 3
E. NOTA
B. 2
D. 5

8. Triangle ABC has side lengths of 20, 22, and 24. Circles are drawn with centers at A, B, and C such that any two circles are externally tangent. Find the sum of the areas of the circles.

A. 371π
C. 395π
E. NOTA
B. 381π
D. 413π

9. If $f(x) = x^3 - 3x^2 + 3x - 2021$, find the area of the region bounded by the graphs of $y = f'(x)$ and $y = f''(x)$.

A. $\frac{8}{3}$
C. $\frac{11}{3}$
E. NOTA
B. $\frac{10}{3}$
D. $\frac{14}{3}$

10. Triangle ABC has an inradius of 2 and a circumradius of 13. $\cos A$ is the average of $\cos B$ and $\cos C$. Find the area of the triangle.

A. 48
C. 52
E. NOTA
B. 50
D. 54

11. Sides DE and EV of triangle DEV are decreasing in length at rates of 2 and 3 units per second. If DEV has constant area, find the rate at which angle E is expanding per second when $DE = 6$, $EV = 9$, and E has measure $\frac{\pi}{6}$.

A. $\frac{2\sqrt{3}}{9}$
C. $\frac{\sqrt{3}}{3}$
E. NOTA
B. $\frac{1}{3}$
D. $\frac{4}{9}$

12. The region bounded by the graph of $y = 16 - x^2$ and the line $y = 1$ is rotated about the x-axis. If the volume of the solid formed is equal to $A\pi\sqrt{15}$, find A.

A. 240
B. 256
C. 270
D. 280
E. NOTA

13. Find the area of the region inside both the graphs of $r = \sin\theta$ and $r = \sin 2\theta$ in the first quadrant.

A. $\frac{\pi}{8} - \frac{3\sqrt{3}}{32}$
B. $\frac{\pi}{8} - \frac{\sqrt{3}}{16}$
C. $\frac{\pi}{8} + \frac{3\sqrt{3}}{32}$
D. $\frac{\pi}{8} + \frac{3\sqrt{3}}{16}$
E. NOTA

14. Which of the following represents the volume of a regular tetrahedron with edge length s?

A. $\frac{s^3\sqrt{2}}{12}$
B. $\frac{s^3\sqrt{3}}{12}$
C. $\frac{s^3\sqrt{2}}{8}$
D. $\frac{s^3\sqrt{3}}{8}$
E. NOTA

15. Croix's elixir is dripping from an inverted cone with radius 6 and height 9 at a constant rate of 12π units per minute. Find the absolute value of the number of units per minute at which the diameter of the surface of the elixir is changing when there are 4π units of elixir left in the cone.

A. 2
B. 4
C. $2\sqrt{3}$
D. $4\sqrt{3}$
E. NOTA

16. A hole of radius 2 is drilled through the center of a sphere with radius 4. Find the volume of the remaining solid.

A. $32\pi\sqrt{3}$
B. $\frac{128\pi\sqrt{3}}{3}$
C. $48\pi\sqrt{3}$
D. $\frac{160\pi\sqrt{3}}{3}$
E. NOTA

17. Evaluate: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3-\left(1+\frac{2k}{n}\right)^2}{n}$.

A. $-\frac{16}{5}$
B. $-\frac{8}{3}$
C. $-\frac{8}{5}$
D. $-\frac{4}{3}$
E. NOTA

18. A point P moves counterclockwise with constant rotational velocity along the unit circle from $(1,0)$ to $(0,1)$ in 5 seconds without retracing any of its path. Let L be the line tangent to the unit circle and passing through P. Find the rate at which the area bounded by L, the unit circle, and the x-axis is changing when L has slope $-\frac{4}{3}$.

A. $\frac{9\pi-\pi^2}{320}$ C. $\frac{8\pi-\pi^2}{90}$ E. NOTA

B. $\frac{9\pi}{320}$ D. $\frac{4\pi}{45}$

19. A rectangle has perimeter P and area A, where both P and A are integers. Given that $\frac{P+4}{P-4} = A - 1$, find the sum of all possible values of P.

A. 8 B. 20 C. 31 D. 32 E. NOTA

x	-6	-4	-2	0	2	4	6	8
$f(x)$	-4	3	2	-7	5	5	-1	6

20. Certain values of $f(x)$ are defined above, Christina and Rohan want to approximate the value of $\int_{-6}^{8} f(x)\,dx$. Christina wants to use a left-hand Riemann sum, and Rohan wants to use a right-hand Riemann sum. If the two students both use rectangles of equal width 2 in their approximations, by how much will Rohan's value exceed Christina's?

A. 8 B. 12 C. 16 D. 20 E. NOTA

21. Find the volume of the solid whose base is bounded by the curve $x^2 + 16y^4 = 16$ and whose cross-sections perpendicular to the x-axis are equilateral triangles.

A. $\frac{\pi\sqrt{3}}{2}$ B. $\pi\sqrt{3}$ C. $2\pi\sqrt{3}$ D. $\frac{8\pi\sqrt{3}}{3}$ E. NOTA

22. The parametric curve given by $x = \cos^3\theta$ and $y = \sin^3\theta$ for $0 \le \theta \le \frac{\pi}{2}$ is rotated about the x-axis. Find the surface area of the figure formed.

A. $\frac{8\pi}{7}$ B. $\frac{6\pi}{5}$ C. $\frac{5\pi}{4}$ D. $\frac{4\pi}{3}$ E. NOTA

23. Find the area of the region bound by the equation $|x + 2y| + |2x - y| = 6$.

A. $\frac{36}{5}$ B. $\frac{36\sqrt{2}}{5}$ C. $\frac{72}{5}$ D. $\frac{36\sqrt{5}}{5}$ E. NOTA

24. A petal is uniformly randomly selected from the graphs of $r = 2\sin 3\theta$ and $r = 3\sin 4\theta$. If the expected value of the area of this petal is $\frac{A\pi}{B}$, find $A + B$.

A. 3 B. 41 C. 57 D. 139 E. NOTA

F. NOTA

25. Find the total area bound by the graph of $y = x + \sin x$ and its inverse between $x = 0$ and $x = 2\pi$.

A. $\pi^2 - 4$
B. 2π
C. $2\pi + 1$
D. $\pi^2 - 2$
E. NOTA

26. A circle with radius 1 centered at $(0,4)$ is rotated about a line tangent to the graph of $y = x^2$. Find the minimum volume of the solid formed.

A. $2\pi^2\sqrt{3}$
B. $\pi^2\sqrt{15}$
C. $4\pi^2$
D. $2\pi^2\sqrt{5}$
E. NOTA

27. Find the area of the triangle whose vertices are the three intersections of the lines $2y - x = 7$, $3x + 5y = 23$, and $4x + 3y = 27$.

A. $\frac{9}{2}$
B. 5
C. $\frac{11}{2}$
D. 6
E. NOTA

28. Let A be the region in the first quadrant bounded by the graph of $y = x\cos x$, the x-axis, and the line $x = \frac{\pi}{2}$. The sum of the volumes of the solids formed when A is rotated over the x- or y-axes can be interpreted as $f(\pi)$, where $f(x)$ is a polynomial with rational coefficients. Find the remainder when $f(12)$ is divided by 19.

A. 2
B. 6
C. 11
D. 14
E. NOTA

29. Find the area of the region bounded by the inner loop of the limaçon $r = 2 + 4\cos\theta$.

A. $4\pi - 6\sqrt{3}$
B. $16\sqrt{3} - 8\pi$
C. $6\sqrt{3} - 2\pi$
D. $6\pi - 8\sqrt{3}$
E. NOTA

30. An ellipse in the complex plane has equation $|z - 1 + 2i| + |z + 3 - i| = 6$. If its area is equal to $\frac{A\pi\sqrt{B}}{C}$ for squarefree B, find $A + B + C$.

A. 11
B. 13
C. 16
D. 20
E. NOTA