Answers: CAABB DDABB ACACC BCBBD ECACD BBCDA

01) This is a geometric series with first term $\frac{2}{3}+\frac{0}{9}+\frac{2}{27}+\frac{1}{81}=\frac{61}{81}$ and common ratio $\frac{1}{81}$, so it has sum $\frac{61/81}{1-1/81}=\frac{61}{80}$. $A+B=141$.

02) Note that $\frac{1}{8}+\frac{1}{16}=\frac{3}{16}$ and $\frac{1}{1-1/16}=\frac{16}{15}$. Dividing the former by the latter yields $\frac{1}{5}$, so $0.\overline{0011}=\frac{1}{5}$. Multiplying by 4 is simply removing the two leading zeroes, so $\frac{4}{5}=0.\overline{1100}$.

03) Use the Method of Finite Differences.

1	6	-12	5	29	-54	-444
5	-18	17	24	-83	-390	
-23	35	7	-107	-307		
58	-28	-114	200			
-86	-86	-86				

The sum of the digits of $|-444|=444$ is 12.

04) Let $a_0=a$ and for all n, $a_n=a+n$. Then $\sqrt{a_1+a_{100}}=\sqrt{2a+101}$ and $\sqrt{a_2+a_3+\cdots+a_{99}}=\sqrt{98a+4949}=7\sqrt{2a+101}$. The expression is equal to $6\sqrt{2a+101}$. a must be a non-negative integer, so $a=10$ and the expression equals 66.

05) By the definition of a Taylor series, $f(x)=7-2(x-3)+\frac{6(x-3)^2}{2}+\frac{12(x-3)^3}{6}=2x^3-15x^2+34x-14$. Plugging in $x=2$ yields 10.

06) Consider $a_{n+2}=\frac{1+\frac{1+a_n}{1-a_n}}{1-\frac{1+a_n}{1-a_n}}=\frac{(1-a_n)+(1+a_n)}{(1-a_n)-(1+a_n)}=\frac{2}{-2a_n}=-\frac{1}{a_n}$. Thus, $a_{n+4}=a_n$ and the sequence is periodic with period 4. $a_{2021}=a_1=2021$.

07) $x^3+6x^2+36x+216$ is asymptotic to $(x+2)^3$ and $x^3+3x^2+9x+27$ is asymptotic to $(x+1)^3$ as x becomes large, so the limit approaches 1.

08) Since the original sum converges, for all sufficiently large $k>K$, $x_k<1$. Then $e^{x_k}-1=x_k+\frac{x_k^2}{2!}+\frac{x_k^3}{3!}+\cdots<x_k+\frac{x_k}{2!}+\frac{x_k}{3!}+\cdots=x_k(e-1)$. $\sum_{k=1}^{K}(e^{x_k}-1)$ has a finite number of terms and thus a finite sum, and since $\sum_{k=K}^{\infty}(e^{x_k}-1)<(e-1)\sum_{k=K}^{\infty}x_k$, the left side converges iff the right side converges.

09) This is the famous Hailstone sequence, subject of the Collatz Conjecture. Writing out terms results in the sequence $\{23,70,35,106,53,160,80,40,20,10,5,16,8,4,2,1,4,2,1,4,2,\ldots\}$. The sequence will repeat forever from here. The sequence repeats every three terms from $n=13$, and since $2021\equiv 2 \bmod 3$, $a_{2021}=2$.

10) Recognizing that $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}=\arctan x$, $\sum_{n=0}^{\infty}\frac{(-3)^{-n}}{2n+1}=\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\left(1/\sqrt{3}\right)^{2n+1}}{2n+1}$. This evaluates to $\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{2\sqrt{3}}$.

11) Simplifying the summand by combining fractions and then taking partial fractions gives $\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}=\sqrt{1+\frac{2n(n+1)+1}{\left(n(n+1)\right)^2}}=\sqrt{\frac{(n(n+1)+1)^2}{\left(n(n+1)\right)^2}}=\frac{n^2+n+1}{n^2+n}=1-\frac{1}{n^2+n}=1+\frac{1}{n}-\frac{1}{n+1}$. Summing, $\sum_{n=1}^{2021}\left(1+\frac{1}{n}-\frac{1}{n+1}\right)$ is a telescoping series equal to $2021+\frac{1}{1}-\frac{1}{2022}=2022-\frac{1}{2022}$.

12) If the decimal form of $\frac{1}{n}$ is terminating, then $n = 2^a 5^b$ for non-negative integers a and b. Note that $2021 = 43 \cdot 47$. For $\frac{2021}{n}$ to be terminating, n must be of the form $2^a 5^b 43^c 47^d$ for non-negative a and b and non-negative integers c and d not greater than 1. The sum of the possible values of $\frac{2021}{n}$ is $\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{1} \sum_{d=0}^{1} \frac{2021}{2^a 5^b 43^c 47^d} = 2021 \sum_{a=0}^{\infty} \frac{1}{2^a} \sum_{b=0}^{\infty} \frac{1}{5^a} \sum_{c=0}^{1} \frac{1}{43^c} \sum_{d=0}^{1} \frac{1}{47^d} = 2021 \cdot 2 \cdot \frac{5}{4} \cdot \frac{44}{43} \cdot \frac{48}{47} = 5280$.

13) The natural logarithm of this expression is $\sum_{n=1}^{\infty} \frac{n}{2^{n+1}}$. If $|x| < 1$ then $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$ and $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$. Thus, $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{(1-1/2)^2} = 4$, so $\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1$ and the product equals e.

14) $\frac{n+48}{n}$ simplifies to $1 + \frac{48}{n}$, which is an integer if $|n|$ is a factor of 48. $48 = 2^4 3$, so it has 10 positive factors and 20 total factors, making for 20 integer elements of a_n.

15) This sum can be calculated manually as $1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 = 330$ or as a case of the Hockey-Stick Identity, where $\binom{11}{4} = 330$.

16) Let $l = \frac{1}{8} + \frac{1}{64} + \frac{1}{216} + \cdots$. Then $l = \frac{1}{8}\left(1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots\right) = \frac{k}{8}$. $h = k - 2l = \frac{3k}{4}$, so $\frac{h}{k} = \frac{3}{4}$.

17) Recognizing this as a Riemann sum, express the sum as $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(4 \cdot \frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right)$. This is equal to $\int_0^4 (4x - x^2)\, dx = 2x^2 - \frac{x^3}{3}\Big]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$. $32 + 3 = 35$.

18) By limit comparison, this series has equivalent convergence with $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}}$. This series is absolutely convergence by the p-series test.

19) Note that with $u = \ln x$, $\int_2^{\infty} \frac{2021}{x \ln^{2021} x}\, dx = \int_{\ln 2}^{\infty} \frac{du}{u^{2021}}$, so the sum is absolutely convergent by the Integral Test.

20) $|x| \le 1$ or the terms would tend towards infinity. If $x = -1$, then the series is convergent by the Alternating Series Test. If $x = 1$, then Cauchy Condensation can be applied, where multiplying the sequence by 2^n and replacing n with 2^n results in $\sum_{n=2}^{\infty} \frac{2^n}{2^n \ln^2 2^n} = \sum_{n=2}^{\infty} \frac{1}{n^2 \ln^2 2}$, which is convergent by limit comparison to the Basel problem. Thus, the interval of convergence is $[-1,1]$.

21) Applying the ratio of consecutive terms yields the quantity $\frac{x^{2n+2}}{x^{2n}} \div \frac{(n+1)^{n+1}}{n^n} = \frac{x^2}{n(1+1/n)^{n+1}}$. Taking the limit at infinity yields $\lim_{n\to\infty} \frac{x^2}{ne}$, which equals 0 for any value of x; the radius of convergence is therefore infinite.

22) The Maclaurin series for $\cos x$ begins $1 - \frac{x^2}{2} + \cdots$, and the Maclaurin series for e^x begins $1 + x + \frac{x^2}{2} + \cdots$. The limit simplifies to $\frac{(x^2/2 - x^4/24 - \cdots)(-x - x^2 - \cdots)}{x^n}$, where the smallest power of x in the numerator is 3. For the limit to exist and be nonzero, n must equal 3.

23) The denominator can be factored using difference of squares, where $4n^2 + 8n + 3 = (2n+2)^2 - 1 = (2n+3)(2n+1)$. Setting up a telescoping series, $\frac{1}{(2n+3)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$. Only the first term will not be cancelled in the summation, so the total sum is $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

24) By the definitions of arithmetic and geometric sequences, $\frac{a_0+2d+5}{a_0+d+1} = \frac{a_0+d+1}{a_0}$. Substituting in $a_0 = 12$, $\frac{17+2d}{13+d} = \frac{13+d}{12}$ and $d^2 + 26d + 169 = 24d + 204$, or $d^2 + 2d - 35 = 0$. The positive solution of this is $d = 5$, so the relevant sequences are $\{12,17,22,27,\ldots\}$ and $\left\{12,18,27,\frac{81}{2},\ldots\right\}$. $b_3 - a_3 = \frac{27}{2}$. $27 + 2 = 29$.

25) Let the sum equal x. Then $\frac{1}{2+\frac{3}{1+\frac{2}{3+x}}} = x$, or $\frac{x+5}{5x+19} = x$. Solving $5x^2 + 18x - 5 = 0$ with the quadratic formula yields $x = \frac{\sqrt{106}-9}{5}$ as the positive solution. $106 + 9 + 5 = 120$.

26) Each sequence consists of the roots of the equation $s_n^3 - (2n+1)s_n^2 + (2n-1)s_n - 1 = 0$ by Vieta's. Inspection yields that $s_n = 1$ is a root of the equation, resulting in $s_n^3 - (2n+1)s_n^2 + (2n-1)s_n - 1 = (s_n - 1)(s_n^2 - 2ns_n - 1) = 0$. The remaining roots are $s_n = n \pm \sqrt{n^2+1}$. The minimum root, a_n, is $n - \sqrt{n^2+1}$. $\lim_{n\to\infty} n\left(n - \sqrt{n^2+1}\right) = \lim_{n\to\infty} -\frac{n}{n+\sqrt{n^2+1}} = \lim_{n\to\infty} -\frac{1}{1+n^{-2}} = -\frac{1}{2}$.

27) The first six harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}$, and after some really nice cancellations, $\frac{49}{20}$. $49 + 20 = 69$.

28) We are trying to find $S = \sum_{n=1}^{\infty} \frac{H_n}{2^n}$. Note that $H_n - H_{n-1} = \frac{1}{n}$. Since $S = \frac{H_1}{2} + \frac{H_2}{4} + \frac{H_3}{8} + \cdots$, $\frac{S}{2} = \frac{H_1}{4} + \frac{H_2}{8} + \frac{H_3}{16} + \cdots$. Subtracting, $\frac{S}{2} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Given that $\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $\frac{S}{2} = \ln 2$ and Carolina will need to pay \$1.39.

29) $\frac{1}{m^2n+mn^2+2mn} = \frac{1}{mn(m+n+2)} = \frac{1}{mn}\int_0^1 x^{m+n+1}\,dx$. $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn}\int_0^1 x^{m+n+1}\,dx = \int_0^1 x \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{x^m x^n}{mn}\,dx = \int_0^1 x \sum_{m=1}^{\infty}\frac{x^m}{m}\sum_{n=1}^{\infty}\frac{x^n}{n}\,dx = \int_0^1 x \ln^2(1-x)dx = \int_0^1 (1-x)\ln^2 x\,dx$. This can be integrated by parts twice, with $u = \ln^2 x$ and $dv = 1 - x$. $\int_0^1 (1-x)\ln^2 x\,dx = \left(x - \frac{x^2}{2}\right)\ln^2 x\Big]_0^1 - \int_0^1 \left(x - \frac{x^2}{2}\right)\cdot\frac{2\ln x}{x}\,dx = \int_0^1 (x-2)\ln x\,dx$. Now set $u = \ln x$ and $dv = x - 2$. $\int_0^1 (x-2)\ln x\,dx = \left(\frac{x^2}{2} - 2x\right)\ln x\Big]_0^1 - \int_0^1 \left(\frac{x^2}{2} - 2x\right)\cdot\frac{1}{x}\,dx = \int_0^1 \left(2 - \frac{x}{2}\right)\,dx = 2 - \frac{1}{4} = \frac{7}{4}$.

30) Let $x = 5$, $y = 4$, and $z = 3$. Splitting the summation and rearranging the order of the sigmas yields $\sum_{a=1}^{x}\sum_{b=1}^{y}\sum_{c=1}^{z}\frac{a}{b} + \sum_{b=1}^{y}\sum_{c=1}^{z}\sum_{a=1}^{x}\frac{b}{c} + \sum_{c=1}^{z}\sum_{a=1}^{x}\sum_{b=1}^{y}\frac{c}{a}$. These sums are all solved the same way. We have $\sum_{a=1}^{x}\sum_{b=1}^{y}\sum_{c=1}^{z}\frac{a}{b} = \sum_{a=1}^{x}\sum_{b=1}^{y}\frac{za}{b} = \sum_{a=1}^{x} zaH_y = zH_y\cdot\frac{x(x+1)}{2}$. Thus, the overall sum is equal to $zH_y\cdot\frac{x(x+1)}{2} + yH_x\cdot\frac{z(z+1)}{2} + xH_z\cdot\frac{y(y+1)}{2}$. Substituting $x = 3$, $y = 4$, and $z = 5$ yields a sum of $5H_4\cdot\frac{3\cdot 4}{2} + 4H_3\cdot\frac{5\cdot 6}{2} + 3H_5\cdot\frac{4\cdot 5}{2} = \frac{125}{2} + 130 + \frac{137}{2} = 241$.