Mu

Sequences and Series

Test #632

Directions:

1. Fill out the top section of the Round 3 Google Form answer sheet and select **Mu- Sequences and Series** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select “E) NOTA” and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like “which of the following could be” or “what is one solution of”, an answer choice providing one of the possibilities is considered to be correct. Do not select “E) NOTA” in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select “E) NOTA” in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms.

NOTA means "None of the Above."

~~~~~~~~ Good luck, and have fun! ~~~~~~~~

1. If $0.\overline{2021}_3 = \frac{A}{B}$ in base 10, find $A + B$.

   A. 87 C. 141 E. NOTA
   B. 139 D. 425

2. Which of the following is equal to $\frac{4}{5}$ in binary?

   A. $0.\overline{1100}$ C. $0.11\overline{01}$ E. NOTA
   B. $0.1\overline{100}$ D. $0.\overline{110}$

3. The graph of a quartic function $f(x)$ contains the points $(-3,1)$, $(-1,6)$, $(1,-12)$, $(3,5)$, and $(5,29)$. Find the sum of the digits of $|f(9)|$.

   A. 12 C. 14 E. NOTA
   B. 13 D. 15

4. $\{a_1, a_2, \dots, a_{100}\}$ is a sequence of consecutive positive integers. Find the minimum integer value of $\sqrt{a_2 + a_3 + \cdots + a_{99}} - \sqrt{a_1 + a_{100}}$.

   A. 54 C. 77 E. NOTA
   B. 66 D. 81

5. For a cubic polynomial $f(x)$, $f(3) = 7$, $f'(3) = -2$, $f''(3) = 6$, and $f'''(3) = 12$. Find $f(2)$.

   A. 9 C. 13 E. NOTA
   B. 10 D. 31

6. A sequence $\{a_n\}$ of real numbers with initial value $a_1 = 2021$ satisfies the recurrence relation $a_{n+1} = \frac{1+a_n}{1-a_n}$ for all $n > 1$. Find a_{2021}.

   A. -2021 C. $\frac{1}{2021}$ E. NOTA
   B. $-\frac{1}{2021}$ D. 2021

7. The sequence $\{a_n\}_{n\geq 1}$ is defined as $a_n = \sqrt[3]{n^3+6n^2+36n+216} - \sqrt[3]{n^3+3n^2+9n+27}$. Compute $\lim_{n\to\infty} a_n$.

A. 0
B. $\sqrt[3]{2}-1$
C. $\sqrt[3]{3}-1$
D. 1
E. NOTA

8. If $\sum_{n=0}^{\infty} x_n$ converges where all x_n are positive reals, then when does $\sum_{n=0}^{\infty} (e^{x_n}-1)$ diverge?

A. Never
B. Only if $\sum_{n=0}^{\infty} \sqrt{x_n}$ diverges
C. Only if $\sum_{n=0}^{\infty} \ln(|1-x_n|)$ diverges
D. Always
E. NOTA

9. Let $\{a_n\}_{n\geq 0}$ be a sequence of positive integers. For all $n \geq 1$, let $a_n = 3a_{n-1}+1$ if a_{n-1} is odd and $a_n = \frac{a_{n-1}}{2}$ if a_{n-1} is even. If $a_0 = 23$, find a_{2021}.

A. 1
B. 2
C. 3
D. 4
E. NOTA

10. Evaluate: $\sum_{n=0}^{\infty} \frac{(-3)^{-n}}{2n+1}$.

A. $\frac{\pi}{6\sqrt{3}}$
B. $\frac{\pi}{2\sqrt{3}}$
C. $\frac{\pi}{3}$
D. $\arctan 3$
E. NOTA

11. Evaluate: $\sum_{n=1}^{2021} \sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}$.

A. $2022 - \frac{1}{2022}$
B. $2022 - \frac{1}{2023}$
C. $2022 - \frac{1}{2022^2}$
D. $2022 - \frac{1}{2022\cdot 2023}$
E. NOTA

12. It is given that the decimal expansion of $\frac{2021}{n}$ is terminating for some positive integer n. Find the sum of all possible values of $\frac{2021}{n}$.

A. 5052.5
B. 5130
C. 5280
D. 5460
E. NOTA

13. Evaluate: $\prod_{n=1}^{\infty} \exp\left(\frac{n}{2^{n+1}}\right)$.

A. e
B. $e\sqrt[e]{e}$
C. $e\sqrt{e}$
D. e^2
E. NOTA

14. The sequence $\{a_n\}_{n\in\mathbb{Z}/0}$ is defined as $a_n = \frac{n+48}{n}$. Find the number of integer elements of a_n.

A. 10 B. 11 C. 20 D. 21 E. NOTA

15. Evaluate: $\sum_{n=3}^{10}\binom{n}{3}$.

A. 165
B. 220
C. 330
D. 495
E. NOTA

16. Let $h = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125} - \frac{1}{216} + \cdots$ and $k = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{216} + \cdots$. Find $\frac{h}{k}$.

A. $\frac{2}{3}$
B. $\frac{3}{4}$
C. $\frac{5}{6}$
D. $\frac{7}{8}$
E. NOTA

17. If $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{64in-64i^2}{n^3} = \frac{J}{B}$, find $J + B$.

A. 19
B. 25
C. 35
D. 41
E. NOTA

18. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{\sqrt{n^5+32}}$.

A. Conditionally convergent
B. Absolutely convergent
C. Divergent
D. Impossible to determine
E. NOTA

19. Determine the convergence of the series $\sum_{n=2}^{\infty} \frac{2021}{n \ln^{2021} n}$.

A. Conditionally convergent
B. Absolutely convergent
C. Divergent
D. Impossible to determine
E. NOTA

20. Find the interval of convergence of the sum $\sum_{n=2}^{\infty}\frac{x^n}{n\ln^2 n}$.

A. $(-\infty,1)$
B. $(-1,1)$
C. $[-1,1)$
D. $[-1,1]$
E. NOTA

21. Find the radius of convergence of the sum $\sum_{n=1}^{\infty}\frac{x^{2n}}{n^n}$.

A. $\frac{1}{2}$
B. 1
C. $\frac{e}{2}$
D. $\sqrt{e}$
E. NOTA

22. Find the unique value of n such that $\lim_{x\to 0}\frac{(1-\cos x)(\cos x-e^x)}{x^n}$ exists and is not equal to 0.

A. 1
B. 2
C. 3
D. 4
E. NOTA

23. Evaluate: $\sum_{n=1}^{\infty}\frac{1}{4n^2+8n+3}$.

A. $\frac{1}{6}$
B. $\frac{2}{9}$
C. $\frac{1}{3}$
D. $\frac{4}{9}$
E. NOTA

24. $\{a_0, a_1, a_2, \ldots\}$ is an increasing arithmetic sequence, and $\{b_0, b_1, b_2, \ldots\}$ is a geometric sequence. If $b_0 = a_0 = 12$, $b_1 = a_1 + 1$, and $b_2 = a_2 + 5$, then $b_3 = a_3 + \frac{C}{T}$. Find $C + T$.

A. 25
B. 27
C. 29
D. 31
E. NOTA

25. Find $A + D + X$, given that A is squarefree and the following:

$$\cfrac{1}{2+\cfrac{3}{1+\cfrac{2}{3+\cfrac{1}{2+\cfrac{3}{1+\cfrac{2}{\ddots}}}}}}=\frac{\sqrt{A}-D}{X}$$

A. 110
B. 115
C. 119
D. 120
E. NOTA

26. $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences of real numbers that satisfy the following system for all $n \geq 1$. Find $\lim_{n\to\infty} na_n$.

$$a_n < b_n < c_n$$
$$a_n + b_n + c_n = 2n + 1$$
$$a_nb_n + b_nc_n + c_na_n = 2n - 1$$
$$a_nb_nc_n = -1$$

A. -1
B. $-\frac{1}{2}$
C. $\frac{1}{2}$
D. 1
E. NOTA

For questions 27-30, define the harmonic numbers $\{H_n\}$ as $H_n = \sum_{k=1}^{n} \frac{1}{k}$. It is well-known that $\lim_{n\to\infty} H_n = \infty$. The integral representation of the harmonic numbers is $H_n = \int_0^1 \frac{1-x^n}{1-x}\,dx$.

27. If $H_6 = \frac{A}{B}$, find $A + B$.

A. 49
B. 69
C. 154
D. 197
E. NOTA

28. Carolina needs to buy an inordinately large number of sardines. Shreya's Seafood is having a sardine sale where the n^{th} sardine you buy costs $\frac{H_n}{2^n}$ cents. As the number of sardines Carolina buys approaches infinity, find the amount of money Carolina will need to pay, rounded to the nearest cent. (You may use the approximation $\ln 2 \approx 0.693$.)

A. \$0.35
B. \$0.69
C. \$1.39
D. \$2.77
E. NOTA

29. Evaluate: $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2n+mn^2+2mn}$.

A. $\frac{4}{3}$
B. $\frac{3}{2}$
C. $\frac{5}{3}$
D. $\frac{7}{4}$
E. NOTA

30. Evaluate: $\sum_{a=1}^{3}\sum_{b=1}^{4}\sum_{c=1}^{5}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$.

A. 241
B. 249
C. 257
D. 261
E. NOTA