Mu

Comprehensive

Test #633

Directions:

1. Fill out the top section of the Round 3 Google Form answer sheet and select **Mu-Comprehensive** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

For all of the following questions, NOTA stands for None Of The Above, and $i = \sqrt{-1}$.

1. Find the area of a triangle with side lengths 6, 8, 10.

   A. 24 B. 25 C. 30 D. 40 E. NOTA

2. Evaluate the integral $\int_0^{\frac{\pi}{2}} (\cos x)(\sin^2 x)\, dx$

   A. 0 B. 1 C. $\frac{1}{2}$ D. $\frac{1}{3}$ E. NOTA

3. Find the sum of the distinct positive factors of 128.

   A. 127 B. 128 C. 255 D. 256 E. NOTA

4. If Daniel rolls a pair of standard 6-sided fair dice, what is the probability that the sum of the two numbers he rolls is a power of 2?

   A. $\frac{1}{2}$ B. $\frac{1}{4}$ C. $\frac{1}{8}$ D. $\frac{1}{16}$ E. NOTA

5. Find the determinant of $\begin{bmatrix} 3 & 4i \\ 4i & 3 \end{bmatrix}$.

   A. 9 B. 16 C. 25 D. 36 E. NOTA

6. Find the sum of the roots of $2x^2 + 16x + 2020 = 0$

   A. 16 B. -16 C. 8 D. -8 E. NOTA

7. What is the value of $\sin\left(\frac{2020\pi}{3}\right)$

   A. $\frac{1}{2}$ B. $\frac{\sqrt{3}}{2}$ C. $-\frac{1}{2}$ D. $-\frac{\sqrt{3}}{2}$ E. NOTA

8. Let r represent a positive root of the equation $ax^2 + bx - c = 0$, where a, b, c are real and positive. In terms of b and c, what is $\lim_{a \to 0} r$?

A. $\frac{b}{c}$ B. $\frac{c}{b}$ C. $-\frac{b}{c}$ D. $-\frac{c}{b}$ E. NOTA

9. Let $f(x)$ be a function representing the volume of a cube with surface area x. Use differentials to approximate $f(10)$, given that a cube with surface area 6 has volume 1.

A. $\frac{3}{2}$ B. $\frac{5}{3}$ C. 2 D. 3 E. NOTA

10. Arnav the Ant is currently sitting at a vertex of a tetrahedron. At every step, he crawls along an edge to an adjacent vertex, giving equal probability to each of his choices. What is the probability that after five steps he has crawled along some edge twice?

A. $\frac{1}{3}$ B. $\frac{16}{27}$ C. $\frac{2}{3}$ D. $\frac{25}{27}$ E. NOTA

For the next 10 questions, we have provided interesting theorems and results from various fields of mathematics. Each pair of questions has to do with the concept that is above them. Make sure to read carefully!

A matrix P is called a **projection matrix** of $\vec{v}$ if for every vector $\vec{u}$,

$$P\vec{u} = \text{proj}_{\vec{v}}\, \vec{u}$$

11. Let P be the projection matrix of $\vec{v} = \langle -1, 2, -2 \rangle$. What is the sum of the elements of P?

A. $\frac{1}{3}$ B. $-\frac{1}{3}$ C. $\frac{1}{9}$ D. $-\frac{1}{9}$ E. NOTA

12. Let P be the projection matrix of $\vec{v} = \langle -5, 3, -2 \rangle$. What is the sum of the elements of $P^2 - P$?

A. $\frac{18}{19}$ B. $\frac{324}{361}$ C. $-\frac{18}{19}$ D. $-\frac{324}{361}$ E. NOTA

For two non-concentric circles, the **radical axis** is the line on which every point has equal power with respect to both circles. For any three circles with non-collinear centers, the three radical axes of each pair of circles concur at a single point, called the **radical center**.

13. Three circles are constructed with centers A, B, C and equal radii. The radical center of the three circles coincides with which point of the non-degenerate triangle ΔABC?
A. Centroid B. Orthocenter C. Incenter D. Circumcenter E. NOTA

14. Three circles are constructed with centers A, B, C and are mutually tangent. The radical center of the three circles coincides with which point of the non-degenerate triangle ΔABC?
A. Centroid B. Orthocenter C. Incenter D. Circumcenter E. NOTA

For a sequence of real numbers $\{a_n\}$, the **generating function** $g(x)$ is given by the series

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

15. Let $f(x)$ be the generating function of the sequence $a_n = 2a_{n-1} - a_{n-2} + 1$, with $a_0 = 0$ and $a_1 = 0$. Find the value of $f\left(\frac{1}{3}\right)$.
A. $\frac{1}{8}$ B. $\frac{3}{8}$ C. $\frac{5}{8}$ D. $\frac{7}{8}$ E. NOTA

16. Let X be a random variable which takes on the value n with probability a_n, where n is a non-negative integer. Let $f(x)$ be the generating function of the sequence of probabilities. Which of the following represents the expected value of X?
A. $\left(f(1)\right)^2$ B. $f(1)$ C. $\left(f'(1)\right)^2$ D. $f'(1)$ E. NOTA

Cauchy-Schwarz Inequality for Integrals states that for any two integrable functions f and g,

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_1^b \left(f(x)\right)^2 dx \int_a^b \left(g(x)\right)^2 dx$$

With equality holding if and only if $f(x) = c \cdot g(x)$ for some constant c.

17. Let R be the region bound by $f(x)$ and the x-axis between the lines $x = 0$ and $x = 10$. Given that the area of R is 10, what is the minimum possible volume of a solid formed when R is rotated around the x-axis?

A. 5π B. 10π C. 15π D. 20π E. NOTA

18. Let R be the region bound by $f(x)$ and the x-axis between the lines $x = 0$ and $x = 3$. The solid formed when R is rotated around the x-axis is 9π. What is the maximum possible volume of a solid formed when R is rotated around the y-axis.

A. $\frac{9\pi}{2}$ B. 9π C. $9\pi\sqrt{2}$ D. 18π E. NOTA

Let $\pi(n)$ denote the number of primes less than or equal to n. Then the following holds:

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln n} = 1$$

19. For some function $f(n)$,

$$\lim_{n\to\infty} \frac{\pi(n \ln n)}{f(n)} = 1$$

Which of the following could $f(n)$ be?

A. n B. $n\sqrt{n}$ C. $n \ln n$ D. $\sqrt{n}\ln n$ E. NOTA

20. Determine the convergence of the pair of series, where p_n denotes the nth prime number.

$$A = \sum_{n=2}^{\infty} \frac{1}{n \cdot \pi(n)}$$

$$B = \sum_{n=2}^{\infty} \frac{1}{p_n \ln n}$$

A. Both A and B converge. B. Both A and B diverge

C. A converges, B diverges. D. A diverges, B converges. E. NOTA

21. The incircle of a certain triangle is tangent to a midsegment of that triangle (a midsegment of a triangle is a segment connecting the midpoints of two sides of that triangle). Two of the sides of the triangle have length 6 and 11. What is the length of the third side, given that it's an integer?

A. 6 B. 7 C. 8 D. 9 E. NOTA

22. Conhur has a fair magical coin with three sides: heads, tails, and arms. Conhur flips his magical coin 10 times. Given that Conhur flipped heads exactly three times, what is the probability that he also flipped arms at least three times?

A. $\frac{1}{2}$ B. $\frac{99}{128}$ C. $\frac{49}{64}$ D. $\frac{25}{32}$ E. NOTA

23. Let ΔABC be a triangle with $AB = 15, BC = 14, AC = 13$. Let O be the circumcenter of ΔABC, and let I be the incenter of ΔABC. Find the area of ΔBOI.

A. 2 B. $\frac{5}{2}$ C. 3 D. $\frac{7}{2}$ E. NOTA

24. What is the most significant digit of 1001^{2020}? (Pick the answer choice that contains the digit.)

A. 1 or 9 B. 2 or 8 C. 3 or 7 D. 4 or 6 E. 5

25. Joanne, Carol and Michelle are located in a plane such that Joanne and Carol are 17 units away, Joanne and Michelle are 25 units away, and Carol and Michelle are 28 units away. At some instant, Joanne, Carol, and Michelle start running toward a mutually agreed meeting point. They all run at the same speed of 1 unit per second, and they reach the meeting point at the same time. How long, in seconds, will it take for the three friends to meet?

A. $\frac{85}{6}$ B. $\frac{99}{7}$ C. 12 D. $\frac{107}{8}$ E. NOTA

26. Call a complex number an **absolute unit** if its absolute value is equal to one. Given absolute units a, b, c with $a + b + c = 1 + i$ and $abc = \frac{3}{5} + \frac{4}{5}i$. What is the value of $a^2 + b^2 + c^2$?

A. $\frac{7}{5} + \frac{9}{5}i$ B. $-\frac{7}{5} + \frac{9}{5}i$ C. $\frac{14}{5} + \frac{8}{5}i$ D. $-\frac{14}{5} + \frac{8}{5}i$ E. NOTA

27. At a certain instant, ΔABC has side lengths $AB = 1, BC = 2, AC = \sqrt{3}$. Point A is moving in the plane of the triangle with respect to segment BC in such a way that $m\angle ABC$ is decreasing at a rate of 1 radian per second, while $m\angle ACB$ is increasing at a rate of 1 radian per second. In units squared per second, how fast is the area of ΔABC increasing at that instant?

A. $\frac{1}{2}$ B. 1 C. $\frac{1}{\pi}$ D. $\frac{1}{2\pi}$ E. NOTA

28. Kev the frog is currently located at $x = 0$ on the number line. After every second passes, he either hops 1 unit in the positive direction, hops 2 units in the positive direction, or he teleports to the point at $x = 4$, each with equal probability. What is the expected value of the number of seconds it will take him to reach his final destination $x = 4$. (Note that he no longer moves once reaching $x = 4$.)

A. $\frac{7}{9}$ B. $\frac{19}{27}$ C. $\frac{20}{9}$ D. $\frac{62}{27}$ E. NOTA

29. Let a, b be positive integers that are solutions to the equation

$$((a - b)^2 + b^2)((a + b)^2 + b^2) = 1 + 4a^2b^2$$

If (a, b) is a solution pair, what is the fourth smallest possible value for $a + b$?

A. 27 B. 28 C. 29 D. 30 E. NOTA

30. For a quadrilateral $ABCD$, let P be the intersection of diagonals AC and BD. Let $WXYZ$ be the centroids of $\Delta ABP, \Delta BCP, \Delta CDP, \Delta DAP$, respectively. Given that $AC = 20, BD = 18$, and the acute angle between them is $60°$. Find the area of $WXYZ$.

A. $20\sqrt{3}$ B. $30\sqrt{3}$ C. $40\sqrt{3}$ D. $60\sqrt{3}$ E. NOTA